

Supplement B:

“Classification of Critical Phenomena having a Parameter-Dependent Renormalization ”

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Ising Hamiltonian for Hanoi Network HN5:

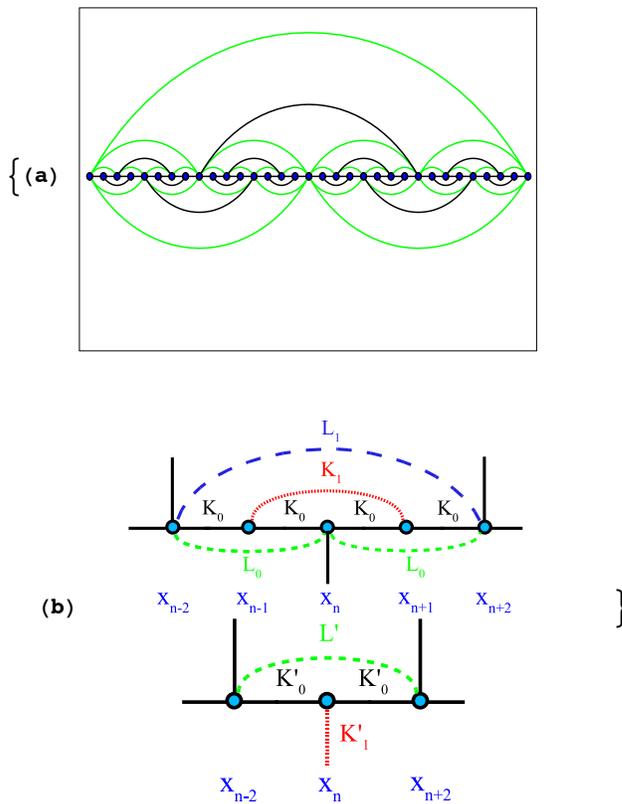


Figure (a) displays the network HN5 (here, with $N=33$ nodes). The Ising Hamiltonian on this network is

$$-\beta\mathcal{H} = K_0(x_0x_1+x_1x_2+\dots+x_{N-1}x_N) + K_1[(x_1x_3+x_5x_7+\dots+x_{N-3}x_{N-1})+(x_2x_6+x_{10}x_{14}+\dots+x_{N-6}x_{N-2})+\dots+(x_{N/4}x_{3N/4})] + L[(x_0x_2+x_2x_4+\dots+x_{N-2}x_N)+(x_0x_4+x_4x_8+\dots+x_{N-4}x_N)+\dots+(x_0x_{N/2}+x_{N/2}x_N)+(x_0x_N)] + H(x_0+\dots+x_N)$$

with raw couplings $K_0 = K_1 = \beta J$, $L = L_0 = L_1 = y\beta J$, and field $H = \beta h$. The parameter $y \in [0, 1]$ allows to define the strength of long-range couplings (marked green in the equation and figure (a)) relative to those in the 1d-backbone of the network, $L = yK_0$.

A single renormalization group (RG) step consists of tracing out every 2nd spin, and we regroup the Hamiltonian into those terms containing all odd-index spins and the rest:

$$-\beta\mathcal{H} = \sum_n (-\beta\mathcal{H}_n) + R(\mathbf{x}_0, \mathbf{x}_2, \dots, \mathbf{x}_{N-2}, \mathbf{x}_N)$$

with the “sectional” Hamiltonians $-\beta\mathcal{H}_n$ for $n=2(2j-1)$, running over all sections $j=1, 2, \dots, N/4$. The top of figure (b) shows the spins and their couplings in each such section of the network. The RG-step consists of tracing out the odd-index spins, \mathbf{x}_{n-1} and \mathbf{x}_{n+1} , in each section. We define “activities” $\mu = \exp\{-2\beta J\}$ and $\eta = \exp\{-\beta h\}$ and parametrize the renormalizable couplings in terms “raw” activities $A_0 = 0$, $A_1 = \mu^2$, $A_2 = \mu^2 \eta$, $A_3 = \eta$, $A_4 = 1$, and $A_5 = 1$, where A_0 becomes the renormalized energy scale, $A_1 = \exp\{-2K_0\}$ and $A_2 = \exp\{-2L\}$ are the renormalized backbone and long-range couplings, $A_3 = \exp\{-H_s\}$ and $A_4 = \exp\{-H_b\}$ are the renormalized site- and bond-magnetizations (initially, $H_s = H$ and $H_b = 0$), and A_5 is the renormalized 3-point magnetization which arises in this RG when the up/down symmetry is broken by the external field. With A_1 representing the coupling strength along the backbone of HN5, its behavior resembles most closely that of κ in our theory.

Then, we can write the sectional Hamiltonian as $Z_0 = \exp\{-\beta\mathcal{H}_n\}$ before the RG-step at any point during the RG recursion as

```
In[1]:= Z0[x0_, x1_, x2_, x3_, x4_] := Exp[-A0] *
  A1 ^ (- (x0 * x1 + x1 * x2 + x2 * x3 + x3 * x4) / 4) * mu ^ (- (x1 * x3) / 2) *
  A2 ^ (- (x0 * x2 + x2 * x4) / 4) * mu ^ (- y (x0 * x4) / 2) *
  A3 ^ (- ((x0 + x1) + (x1 + x2) + (x2 + x3) + (x3 + x4)) / 4) *
  A4 ^ (- ((x0 + x2) + (x2 + x4)) / 4) *
  A5 ^ (- (x0 * x1 * x2 + x2 * x3 * x4) / 2)
```

Notice the explicit dependence on the raw activity μ originating with the long-range couplings K_1 that do not get renormalized, see Figure (b) above.

Obtaining the RG-flow:

The RG-step consists of tracing out the odd-index spins in Z_0 from before the RG-step, and express the result in a function Z_1 after the RG-step that is identical in shape (see bottom of figure (b)), i.e.,

```
In[2]:= Z1[x0_, x2_, x4_] := Exp[-B0 / 2] *
  B1 ^ (- (x0 * x2 + x2 * x4) / 4) *
  B2 ^ (- (x0 * x4) / 4) *
  B3 ^ (- ((x0 + x2) + (x2 + x4)) / 4) *
  B4 ^ (- (x0 * x4) / 4) *
  B5 ^ (- (x0 * x2 * x4) / 2)
```

where the B_i are the newly renormalized activities. To this end, we require $Z_1[\mathbf{x}_{n-2}, \mathbf{x}_n, \mathbf{x}_{n+2}] = \sum_{\mathbf{x}_{n-1}} \sum_{\mathbf{x}_{n+1}} Z_0[\mathbf{x}_{n-2}, \mathbf{x}_{n-1}, \mathbf{x}_n, \mathbf{x}_{n+1}, \mathbf{x}_{n+2}]$:

```
In[3]:= TrZ0[x0_, x2_, x4_] := Sum[Sum[Z0[x0, x1, x2, x3, x4], {x1, -1, 1, 2}], {x3, -1, 1, 2}]
```

Hence, both sides have to be equal for all combinations of $\mathbf{x}_{n-2}, \mathbf{x}_n, \mathbf{x}_{n+2} = + / - 1$,

```

In[4]:= i = 0;
For[xn-2 = -1, xn-2 ≤ 1, xn-2 += 2, For[xn = -1, xn ≤ 1, xn += 2, For[xn+2 = -1, xn+2 ≤ 1, xn+2 += 2,
  leq[i] = Z1[xn-2, xn, xn+2];
  req[i++] = Factor[TrZ0[xn-2, xn, xn+2]];
]]]
leq[0] == req[0]
leq[1] == req[1]
leq[2] == req[2]
leq[3] == req[3]
leq[4] == req[4]
leq[5] == req[5]
leq[6] == req[6]
leq[7] == req[7]

```

$$\text{Out[6]= } \frac{e^{-\frac{B_0}{2}} B_3 \sqrt{B_4} \sqrt{B_5}}{\sqrt{B_1} B_2^{1/4}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} - \frac{y}{2}} A_4 (A_1^2 + 2 \mu A_1 A_3 A_5 + A_3^2 A_5^2)}{A_1 \sqrt{A_2} A_5}$$

$$\text{Out[7]= } \frac{e^{-\frac{B_0}{2}} B_2^{1/4} \sqrt{B_3}}{\sqrt{B_5}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} + \frac{y}{2}} \sqrt{A_4} (\mu A_1 A_3 + A_1 A_5 + A_3^2 A_5 + \mu A_3 A_5^2)}{\sqrt{A_1} \sqrt{A_3} A_5}$$

$$\text{Out[8]= } \frac{e^{-\frac{B_0}{2}} \sqrt{B_1} \sqrt{B_4}}{B_2^{1/4} \sqrt{B_5}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} - \frac{y}{2}} \sqrt{A_2} (A_3^2 + 2 \mu A_3 A_5 + A_5^2)}{A_3 A_5}$$

$$\text{Out[9]= } \frac{e^{-\frac{B_0}{2}} B_2^{1/4} \sqrt{B_5}}{\sqrt{B_3}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} + \frac{y}{2}} (\mu A_3 + A_5 + A_1 A_3^2 A_5 + \mu A_1 A_3 A_5^2)}{\sqrt{A_1} A_3^{3/2} \sqrt{A_4} A_5}$$

$$\text{Out[10]= } \frac{e^{-\frac{B_0}{2}} B_2^{1/4} \sqrt{B_3}}{\sqrt{B_5}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} + \frac{y}{2}} \sqrt{A_4} (\mu A_1 A_3 + A_1 A_5 + A_3^2 A_5 + \mu A_3 A_5^2)}{\sqrt{A_1} \sqrt{A_3} A_5}$$

$$\text{Out[11]= } \frac{e^{-\frac{B_0}{2}} \sqrt{B_1} \sqrt{B_5}}{B_2^{1/4} \sqrt{B_4}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} - \frac{y}{2}} \sqrt{A_2} (A_3^2 + 2 \mu A_3 A_5 + A_5^2)}{A_3 A_5}$$

$$\text{Out[12]= } \frac{e^{-\frac{B_0}{2}} B_2^{1/4} \sqrt{B_5}}{\sqrt{B_3}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} + \frac{y}{2}} (\mu A_3 + A_5 + A_1 A_3^2 A_5 + \mu A_1 A_3 A_5^2)}{\sqrt{A_1} A_3^{3/2} \sqrt{A_4} A_5}$$

$$\text{Out[13]= } \frac{e^{-\frac{B_0}{2}}}{\sqrt{B_1} B_2^{1/4} B_3 \sqrt{B_4} \sqrt{B_5}} == \frac{e^{-A_0} \mu^{-\frac{1}{2} - \frac{y}{2}} (1 + 2 \mu A_1 A_3 A_5 + A_1^2 A_3^2 A_5^2)}{A_1 \sqrt{A_2} A_3^2 A_4 A_5}$$

Due to the remaining symmetries, not all of these relations are independent. In fact, since there are 6 renormalizable couplings, there are also six independent relations. (To simplify the task for *Mathematica*, it is advisable to eliminate unnecessary dependences before solving!)

```

In[14]:= SB[0] = Expand[Solve[Log[leq[0] * leq[1] * leq[2] * leq[3] * leq[4] * leq[5] * leq[6] * leq[7]] ==
  Log[Simplify[req[0] * req[1] * req[2] * req[3] * req[4] *
    req[5] * req[6] * req[7] * Exp[8 * A0]]] - 8 * A0, B0]]][[1]]
SB[1] = PowerExpand[Simplify[Solve[Factor[(leq[2] * leq[5]) / (leq[0] * leq[7])] ==
  Factor[(req[2] * req[5]) / (req[0] * req[7])], B1]]][[2]]]
SB[2] = PowerExpand[Simplify[Solve[Factor[(leq[1]^2 * leq[3]^2) /
  (leq[2] * leq[5] * leq[0] * leq[7])] ==
  Factor[(req[1]^2 * req[3]^2) / (req[2] * req[5] * req[0] * req[7])], B2]]][[2]]]
SB[3] = PowerExpand[Simplify[Solve[Factor[(leq[1]^2 * leq[0] * leq[5]) /
  (leq[3]^2 * leq[7] * leq[2])] ==
  Factor[(req[1]^2 * req[0] * req[5]) / (req[3]^2 * req[7] * req[2])], B3]]][[4]]]
SB[4] = PowerExpand[Simplify[Solve[Factor[(leq[3]^2 * leq[2]^3 * leq[0]) /
  (leq[5]^3 * leq[1]^2 * leq[7])] ==
  Factor[(req[3]^2 * req[2]^3 * req[0]) / (req[5]^3 * req[1]^2 * req[7])], B4]]][[4]]]
SB[5] = PowerExpand[Simplify[Solve[Factor[(leq[3]^2 * leq[0] * leq[5]) /
  (leq[1]^2 * leq[7] * leq[2])] ==
  Factor[(req[3]^2 * req[0] * req[5]) / (req[1]^2 * req[7] * req[2])], B5]]][[4]]]

RGh→0 = Table[Part[SB[n], 1], {n, 0, 5}];

```

$$\text{Out[14]} = \left\{ B_0 \rightarrow \text{Log}[\mu] + \text{Log}[A_1] + 2 \text{Log}[A_3] + 2 \text{Log}[A_5] - \frac{1}{2} \text{Log}[A_3^2 + 2 \mu A_3 A_5 + A_5^2] - \right. \\
\left. \frac{1}{4} \text{Log}[A_1^2 + 2 \mu A_1 A_3 A_5 + A_3^2 A_5^2] - \frac{1}{4} \text{Log}[1 + 2 \mu A_1 A_3 A_5 + A_1^2 A_3^2 A_5^2] - \right. \\
\left. \frac{1}{2} \text{Log}[A_1 (\mu A_3 + A_5) + A_3 A_5 (A_3 + \mu A_5)] - \frac{1}{2} \text{Log}[A_5 + A_1 A_3^2 A_5 + A_3 (\mu + \mu A_1 A_5^2)] + 2 A_0 \right\}$$

$$\text{Out[15]} = \left\{ B_1 \rightarrow \left(A_1 A_2 (A_3^2 + 2 \mu A_3 A_5 + A_5^2) \right) / \left(\sqrt{(A_3^2 A_5^2 + A_1^4 A_3^2 A_5^2 + 2 \mu A_1 A_3 A_5 (1 + A_3^2 A_5^2) + 2 \mu A_1^3 A_3 A_5 (1 + A_3^2 A_5^2) + A_1^2 (1 + 4 \mu^2 A_3^2 A_5^2 + A_3^4 A_5^4))} \right) \right\}$$

$$\text{Out[16]} = \left\{ B_2 \rightarrow \left(\mu^{2y} (A_1 (\mu A_3 + A_5) + A_3 A_5 (A_3 + \mu A_5)) (A_5 + A_1 A_3^2 A_5 + A_3 (\mu + \mu A_1 A_5^2)) \right) / \left(\left(A_3^2 + 2 \mu A_3 A_5 + A_5^2 \right) \sqrt{A_1^2 + 2 \mu A_1 A_3 A_5 + A_3^2 A_5^2} \sqrt{1 + 2 \mu A_1 A_3 A_5 + A_1^2 A_3^2 A_5^2} \right) \right\}$$

$$\text{Out[17]} = \left\{ B_3 \rightarrow \left(A_3 A_4 (A_1^2 + 2 \mu A_1 A_3 A_5 + A_3^2 A_5^2)^{1/4} \sqrt{A_1 (\mu A_3 + A_5) + A_3 A_5 (A_3 + \mu A_5)} \right) / \left(\left(1 + 2 \mu A_1 A_3 A_5 + A_1^2 A_3^2 A_5^2 \right)^{1/4} \sqrt{A_5 + A_1 A_3^2 A_5 + A_3 (\mu + \mu A_1 A_5^2)} \right) \right\}$$

$$\text{Out[18]} = \left\{ B_4 \rightarrow \left((A_1^2 + 2 \mu A_1 A_3 A_5 + A_3^2 A_5^2)^{1/4} \sqrt{A_5 + A_1 A_3^2 A_5 + A_3 (\mu + \mu A_1 A_5^2)} \right) / \left(\left(1 + 2 \mu A_1 A_3 A_5 + A_1^2 A_3^2 A_5^2 \right)^{1/4} \sqrt{A_1 (\mu A_3 + A_5) + A_3 A_5 (A_3 + \mu A_5)} \right) \right\}$$

$$\text{Out[19]} = \left\{ B_5 \rightarrow \left((A_1^2 + 2 \mu A_1 A_3 A_5 + A_3^2 A_5^2)^{1/4} \sqrt{A_5 + A_1 A_3^2 A_5 + A_3 (\mu + \mu A_1 A_5^2)} \right) / \left(\left(1 + 2 \mu A_1 A_3 A_5 + A_1^2 A_3^2 A_5^2 \right)^{1/4} \sqrt{A_1 (\mu A_3 + A_5) + A_3 A_5 (A_3 + \mu A_5)} \right) \right\}$$

These are the RG-flow equations. Again, notice the dependence on the temperature β through the explicit occurrence of μ .

As mentioned above, the RG-flow is initiated through the raw couplings via

$$\text{In[21]} = \mathbf{SIC} = \{B_0 \rightarrow 0, B_1 \rightarrow \mu^2, B_2 \rightarrow \mu^{2y}, B_3 \rightarrow \eta, B_4 \rightarrow 1, B_5 \rightarrow 1\}$$

$$\text{Out[21]} = \{B_0 \rightarrow 0, B_1 \rightarrow \mu^2, B_2 \rightarrow \mu^{2y}, B_3 \rightarrow \eta, B_4 \rightarrow 1, B_5 \rightarrow 1\}$$

Fixed Points at h=0:

First, we consider the case $h=0$ ($\eta=1$), in which case the RG-flow for the activities A_3, A_4 , and A_5 become trivial:

In[22]= **RG_{h0} = FullSimplify[RG_{h→0} /. η → 1 /. A₃ → 1 /. A₄ → 1 /. A₅ → 1, Assumptions → μ > 0 && γ > 0 && A₁ > 0 && A₂ > 0]**

$$\text{Out[22]= } \left\{ B_0 \rightarrow \text{Log}[\mu] + \text{Log}[A_1] + \frac{1}{2} \text{Log}\left[\frac{1}{2(1+\mu)^3(1+A_1)^2(1+2\mu A_1+A_1^2)}\right] + 2A_0, \right. \\ \left. B_1 \rightarrow \frac{2(1+\mu)A_1A_2}{1+2\mu A_1+A_1^2}, B_2 \rightarrow \frac{\mu^{2\gamma}(1+\mu)(1+A_1)^2}{2(1+2\mu A_1+A_1^2)}, B_3 \rightarrow 1, B_4 \rightarrow 1, B_5 \rightarrow 1 \right\}$$

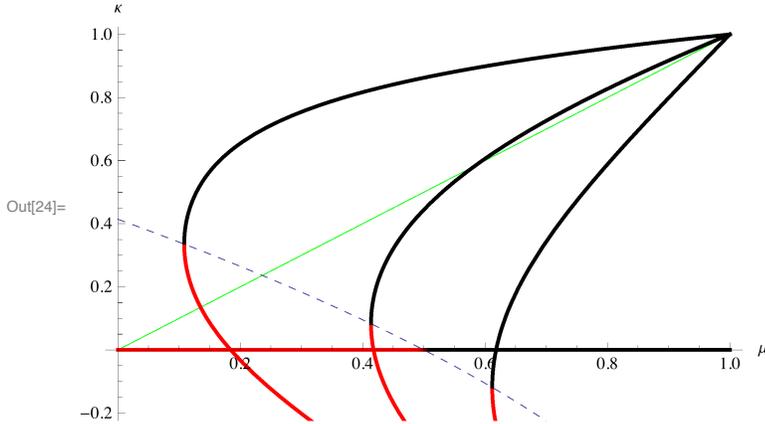
The recursions for A_1 and A_2 correspond to those for κ and λ in Eq. (19) of Ref. [Boettcher&Brunson, 2011]. We obtain the fixed points by equating $B_i = A_i$ in the RG-flow:

In[23]= **FP = FullSimplify[Solve[({B₁, B₂} /. RG_{h0}) == {A₁, A₂}, {A₁, A₂}], Assumptions → μ > 0 && γ > 0]**

$$\text{Out[23]= } \left\{ \left\{ A_1 \rightarrow 0, A_2 \rightarrow \frac{1}{2} \mu^{2\gamma} (1+\mu) \right\}, \left\{ A_1 \rightarrow \frac{1}{2} \left(\mu^\gamma + \mu(-2+\mu^\gamma) - \sqrt{-4+4\mu^\gamma(1+\mu) + (-2\mu+\mu^\gamma(1+\mu))^2} \right), \right. \right. \\ \left. A_2 \rightarrow \frac{1}{4} \mu^\gamma \left(2+\mu^\gamma + \mu(-2+\mu^\gamma) - \sqrt{-4+4\mu^2+\mu^{2\gamma}(1+\mu)^2-4\mu^\gamma(-1+\mu^2)} \right) \right\}, \\ \left\{ A_1 \rightarrow \frac{1}{2} \left(\mu^\gamma + \mu(-2+\mu^\gamma) + \sqrt{-4+4\mu^\gamma(1+\mu) + (-2\mu+\mu^\gamma(1+\mu))^2} \right), \right. \\ \left. A_2 \rightarrow \frac{1}{4} \mu^\gamma \left(2+\mu^\gamma + \mu(-2+\mu^\gamma) + \sqrt{-4+4\mu^2+\mu^{2\gamma}(1+\mu)^2-4\mu^\gamma(-1+\mu^2)} \right) \right\}, \\ \left\{ A_1 \rightarrow \frac{1}{2} \left(-2\mu - \mu^\gamma(1+\mu) - \sqrt{-4+4\mu^2+\mu^{2\gamma}(1+\mu)^2+4\mu^\gamma(-1+\mu^2)} \right), \right. \\ \left. A_2 \rightarrow \frac{1}{4} \mu^\gamma \left(-2+\mu^\gamma + \mu(2+\mu^\gamma) + \sqrt{-4+4\mu^2+\mu^{2\gamma}(1+\mu)^2+4\mu^\gamma(-1+\mu^2)} \right) \right\}, \\ \left\{ A_1 \rightarrow \frac{1}{2} \left(-2\mu - \mu^\gamma(1+\mu) + \sqrt{-4+4\mu^2+\mu^{2\gamma}(1+\mu)^2+4\mu^\gamma(-1+\mu^2)} \right), \right. \\ \left. A_2 \rightarrow \frac{1}{4} \mu^\gamma \left(-2+\mu^\gamma + \mu(2+\mu^\gamma) - \sqrt{-4+4\mu^2+\mu^{2\gamma}(1+\mu)^2+4\mu^\gamma(-1+\mu^2)} \right) \right\} \right\}$$

A plot of the FP for the backbone activity A_1 ($= \kappa$) as a function of μ for various values of γ exhibits exactly the same behavior as our model, see Fig. 1(b-d) for comparison:

```
In[24]:= Show[
  Plot[μ, {μ, 0, 1}, PlotStyle → Green],
  Plot[-1 + √(2 - 2 μ), {μ, 0, 1}, PlotStyle → {Dashed}],
  Plot[{A1 /. (FP /. y → .1), A1 /. (FP /. y → .4), A1 /. (FP /. y → 1)}, {μ, 0, 1},
  PlotStyle → Thick, ColorFunction → Function[{a, b}, If[b < -1 + √(2 - 2 a), Red, Black]],
  ColorFunctionScaling → False]
, PlotRange → {{0, 1}, {-0.2, 1}}, AxesLabel → {"μ", "κ"}]
```



Notice that we have included in the plot (as a dashed line) the function for the location of the branch-point,

$A_1(\mu_B) = \kappa_B(\mu_B) = -1 + \sqrt{2 - 2\mu_B}$, which results from the vanishing of the discriminant in the FP of A_1 . To this end, we define a general procedure which evaluates any input quantity as a function of μ_B ...

```
In[25]:= OnMuB[exp_] := Block[{res, Defz, Solz, z, ψ, τ},
  Defz = Collect[(-4 + 4 μy (1 + μ) + (-2 μ + μy (1 + μ))2) /. μy (1 + μ) → z, z];
  Solz = Simplify[Solve[Defz == 0, z]][[2]];
  res = Factor[Simplify[exp /. y → Log[z / (1 + μ)] / Log[μ], Assumptions → z > 0 && μ > 0]];
  res = FullSimplify[res /. Solz, Assumptions → μ < 1 && μ > 0];
  res = FullSimplify[res /. μ → 1 - τ2, Assumptions → τ > 0];
  res = FullSimplify[res /. τ → ψ + Sqrt[2], Assumptions → ψ > 0];
  res = FullSimplify[res /. ψ → Sqrt[1 - μ] - Sqrt[2], Assumptions → μ > 0];
  Return[res /. μ → μB];
]
```

...and use it to determine $\kappa_B(\mu_B)$ from the upper branch of fixed points (FP), $A_1 = \kappa_+(\mu) \rightarrow \kappa_B(\mu_B)$ for $\mu \rightarrow \mu_B$:

```
In[26]:= OnMuB[A1 /. FP[[3]]]
```

```
Out[26]= -1 + √(2 - 2 μB)
```

Now we evaluate the eigenvalues λ as in Eq. (7) from the Jacobian defined in Eq. (3) at $h=0$:

```
In[27]:= MatrixForm[
  (R')h0 = Table[FullSimplify[D[(Bi /. RGh=0) /. η → 1 /. A3 → 1 /. A4 → 1 /. A5 → 1], Aj],
  Assumptions → μ > 0 && y > 0 && A1 > 0 && A2 > 0], {j, 1, 2}, {i, 1, 2}]]
```

```
Out[27]//MatrixForm=
```

$$\begin{pmatrix} -\frac{2(1+\mu)(-1+A_1^2)A_2}{(1+2\mu A_1+A_1^2)^2} & \frac{\mu^{2y}(-1+\mu^2)(-1+A_1^2)}{(1+2\mu A_1+A_1^2)^2} \\ \frac{2(1+\mu)A_1}{1+2\mu A_1+A_1^2} & 0 \end{pmatrix}$$

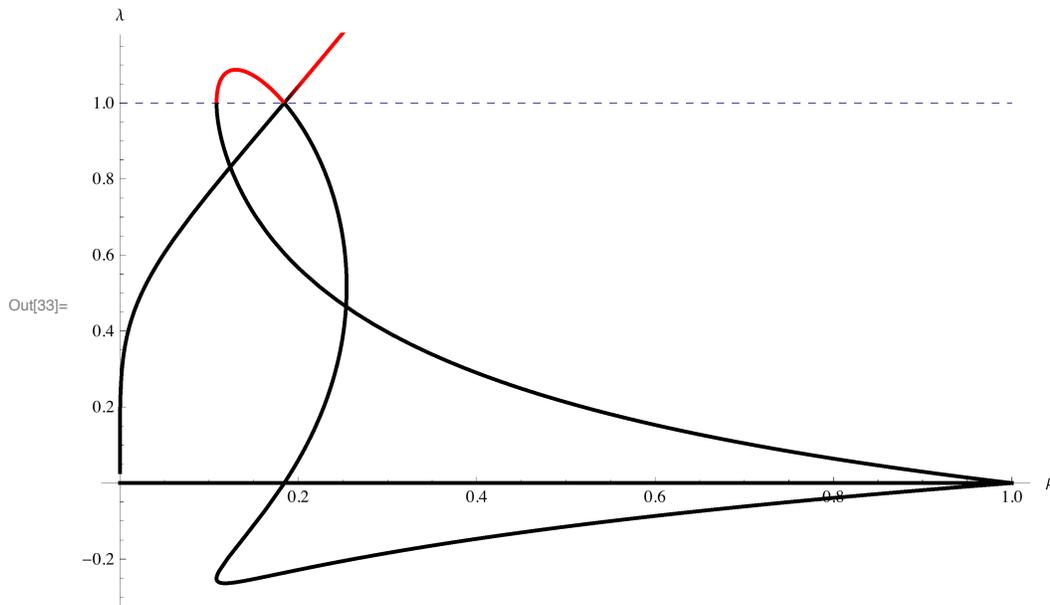
with eigenvalues (very long outputs have been suppressed!)

```
In[28]:= EVh0 = Eigenvalues[(R')h0];
λ0 = Simplify[(EVh0 /. FP[[1]])], Assumptions → μ > 0 && y > 0]
λ+ = Simplify[(EVh0 /. FP[[2]])], Assumptions → μ > 0 && y > 0];
λ- = Simplify[(EVh0 /. FP[[3]])], Assumptions → μ > 0 && y > 0];
```

```
Out[29]:= {μ2y (1 + μ)2, 0}
```

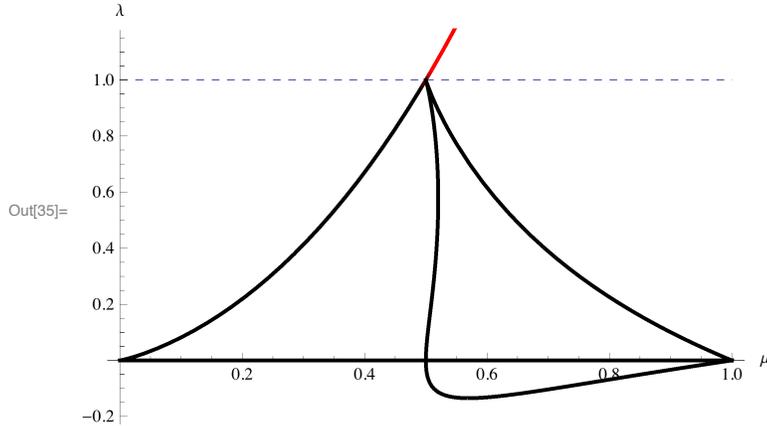
From those relations, we can reproduce the eigenvalue-plot in Fig. 2(b) for a branch point in the physical regime, $\kappa_B(\mu_B) > 0$:

```
In[32]:= Ev = {λ0, λ+, λ-} /. y → 0.1;
Show[
  Plot[1, {μ, 0, 1}, PlotStyle → Dashed],
  Plot[Ev, {μ, 0, 1}, PlotStyle → Thick,
    ColorFunction → Function[{a, b}, If[b > 1, Red, Black]], ColorFunctionScaling → False],
  PlotRange → {{0, 1}, {-0.3, 1.15}}, AxesLabel → {"μ", "λ"}]
```



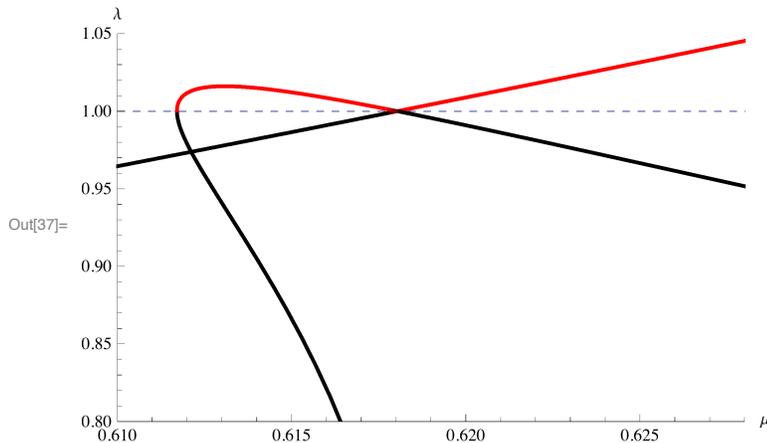
...or the corresponding plot at the critical value of $y_c = \log_2(3/2)$ when $\kappa_B(\mu_B) = 0$:

```
In[34]:= Ev = {λ0, λ+, λ-} /. y → Log[3 / 2] / Log[2];
Show[
  Plot[1, {μ, 0, 1}, PlotStyle → Dashed],
  Plot[Ev, {μ, 0, 1}, PlotStyle → Thick,
    ColorFunction → Function[{a, b}, If[b > 1, Red, Black]], ColorFunctionScaling → False],
  PlotRange → {{0, 1}, {-0.2, 1.15}}, AxesLabel → {"μ", "λ"}]
```



...or for $y=1$ (somewhat enlarged) when the branch point is below the physical regime, $\kappa_B(\mu_B) < 0$, and λ_+ , λ_- have switched order:

```
In[36]:= Ev = {λ0, λ+, λ-} /. y → 1;
Show[
  Plot[1, {μ, 0.61, 0.63}, PlotRange → {{0.61, 0.628}, {.8, 1.05}}, PlotStyle → Dashed],
  Plot[Ev, {μ, 0.61, 0.63}, PlotRange → {{0.61, 0.628}, {.8, 1.05}}, PlotStyle → Thick,
    ColorFunction → Function[{a, b}, If[b > 1, Red, Black]], ColorFunctionScaling → False],
  AxesLabel → {"μ", "λ"}]
```



Fixed Points at $h>0$:

Inducing a magnetization for a small external field $h \rightarrow 0$ breaks the up/down (\mathbb{Z}_2)-symmetry and requires three new parameters, the activities A_3 , A_4 , and A_5 above, to be renormalized, and we arrive at the extended RG-flow, $RG_{h \rightarrow 0}$ above. For any $h > 0$, the RG-flow evolves to the stable strong-field fixed point with $A_3 = 0$; the behavior for small external field $h \rightarrow 0$ is governed by the unstable fixed point $A_3 = A_4 = A_5 = 1$ (leaving A_1 , A_2 unevaluated for now), where we obtain the extended Jacobian matrix

```
In[38]:= MatrixForm[ (R')_{h→0} = Table[FullSimplify[D[ (B_i / RG_{h→0}), A_j] /. A_3 → 1 /. A_4 → 1 /. A_5 → 1,
Assumptions → μ > 0 && γ > 0 && A_1 > 0 && A_2 > 0], {j, 1, 5}, {i, 1, 5}]]
```

```
Out[38]//MatrixForm=
```

$$\begin{pmatrix} -\frac{2(1+\mu)(-1+A_1^2)A_2}{(1+2\mu A_1+A_1^2)^2} & \frac{\mu^2 \gamma (-1+\mu^2)(-1+A_1^2)}{(1+2\mu A_1+A_1^2)^2} & 0 & 0 & 0 \\ \frac{2(1+\mu)A_1}{1+2\mu A_1+A_1^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{1}{1+\mu} + \frac{2}{(1+\mu)(1+A_1)} + \frac{1+\mu A_1}{1+2\mu A_1+A_1^2} & -\frac{(-1+\mu)(-1+A_1)^3}{2(1+\mu)(1+A_1)(1+2\mu A_1+A_1^2)} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} + \frac{1}{1+\mu} + \frac{2\mu}{(1+\mu)(1+A_1)} + \frac{1+\mu A_1}{1+2\mu A_1+A_1^2} & \frac{1}{2} - \frac{1}{1+\mu} - \frac{2\mu}{(1+\mu)(1+A_1)} + \frac{1+\mu A_1}{1+2\mu A_1+A_1^2} & \frac{1}{2} - \frac{1}{1+\mu} \end{pmatrix}$$

The upper block for A_1, A_2 is identical to the 2x2 matrix above, and is decoupled from the field-dependent lower block. Just focusing on the lower block:

```
In[39]:= MatrixForm[ (R')_{h→0} = Table[FullSimplify[D[ (B_i / RG_{h→0}), A_j] /. A_3 → 1 /. A_4 → 1 /. A_5 → 1,
Assumptions → μ > 0 && γ > 0 && A_1 > 0 && A_2 > 0], {j, 3, 5}, {i, 3, 5}]]
```

```
Out[39]//MatrixForm=
```

$$\begin{pmatrix} \frac{1}{2} - \frac{1}{1+\mu} + \frac{2}{(1+\mu)(1+A_1)} + \frac{1+\mu A_1}{1+2\mu A_1+A_1^2} & -\frac{(-1+\mu)(-1+A_1)^3}{2(1+\mu)(1+A_1)(1+2\mu A_1+A_1^2)} & -\frac{(-1+\mu)(-1+A_1)^3}{2(1+\mu)(1+A_1)(1+2\mu A_1+A_1^2)} \\ 1 & 0 & 0 \\ -\frac{3}{2} + \frac{1}{1+\mu} + \frac{2\mu}{(1+\mu)(1+A_1)} + \frac{1+\mu A_1}{1+2\mu A_1+A_1^2} & \frac{1}{2} - \frac{1}{1+\mu} - \frac{2\mu}{(1+\mu)(1+A_1)} + \frac{1+\mu A_1}{1+2\mu A_1+A_1^2} & \frac{1}{2} - \frac{1}{1+\mu} - \frac{2\mu}{(1+\mu)(1+A_1)} + \frac{1+\mu A_1}{1+2\mu A_1+A_1^2} \end{pmatrix}$$

...provides the eigenvalues:

```
In[40]:= EV_{h→0} = FullSimplify[Eigenvalues[ (R')_{h→0}], Assumptions → μ > 0 && A_1 > 0]
```

```
Out[40]= {0, \frac{(-1+\mu)(-1+A_1)}{(1+\mu)(1+A_1)}, \frac{2+2\mu A_1}{1+2\mu A_1+A_1^2}}
```

For physical μ and fixed points of A_1 , the dominant eigenvalue is

```
In[41]:= λ_h = (EV_{h→0})[[3]]
```

```
Out[41]= \frac{2+2\mu A_1}{1+2\mu A_1+A_1^2}
```

Note that $\lambda_h \leq 2$, and in particular, $\lambda_h = 2$ for the fixed point with $A_1 = \kappa_+ (\mu_C) = 0$. For $A_1 > 0$, it is

```
In[42]:= Series[λ_h, {A_1, 0, 1}]
```

```
Out[42]= 2 - 2\mu A_1 + O[A_1]^2
```

for any y , in accordance with Eq. (12).

Determination of magnetic exponent β for HN5 at $y=0.1$:

We can now use the above quantities to determine, for example, the magnetic exponent of the Ising model on HN5 at $y=0.1$, that was given as $\beta=0.205\dots$ in Fig. 3(a). To this end, we first have to locate the critical point μ_C from the raw couplings with the unstable FP-branch. Unlike for the model, in HN5 this is not merely provided by the intersection $\kappa_0 (\mu_C) = \kappa_- (\mu_C)$, since HN5 possesses *two* renormalizable couplings, A_1 and A_2 , that form an unstable FB-manifold. As this FP is unstable, we proceed with a “shooting” and bi-sectioning approach where we evolve repeatedly the RG-flow from the raw couplings at some value of μ and either increase or decrease μ with ever smaller increments, depending on whether the RG-flow has reached a stable strong- or weak-coupling FP. After several iterations of this procedure, the RG-flow will remain near the unstable FP for many recursions, indicating that we are close to μ_C .

We recall that the RG-recursions for A_1 and A_2 :

$$\{B_1, B_2\} / \cdot RG_{h0}$$

$$\left\{ \frac{2(1+\mu)A_1A_2}{1+2\mu A_1+A_1^2}, \frac{\mu^{2y}(1+\mu)(1+A_1)^2}{2(1+2\mu A_1+A_1^2)} \right\}$$

In the following, we have already iterated by hand the “shooting” and bi-sectioning procedure and found a good approximation for $\mu_C = 0.165805 \dots$ at $y=0.1$, confirmed by the fact that the RG-flow remains near the unstable FP for many recursions (only $i=10$ are shown):

```
In[45]:= y = .1
μ = 0.165805160187
A1 = μ2
A2 = μ(2 y)
For[i = 0, i < 10, i++, Print[i, " ", {A1, A2} = {B1, B2} /. RGh0]]
```

Out[45]= 0.1

Out[46]= 0.165805

Out[47]= 0.0274914

Out[48]= 0.698103

```
0 {0.0443104, 0.425408}
1 {0.0432307, 0.436516}
2 {0.0432979, 0.435808}
3 {0.0432937, 0.435852}
4 {0.043294, 0.435849}
5 {0.043294, 0.435849}
6 {0.043294, 0.435849}
7 {0.043294, 0.435849}
8 {0.043294, 0.435849}
9 {0.043294, 0.435849}
```

This value of $\mu_C = 0.165805 \dots$ corresponds to a critical temperature of

```
In[50]:= Tc = -2 / Log[μ]
```

Out[50]= 1.113

In particular, at this μ_C , we find the temperature exponent $y_t = \log_2[\lambda_- (\mu_C)]$:

```
In[51]:= yt = Log[EVh0[[1]]] / Log[2]
```

Out[51]= 0.0629237

and the magnetic exponent $y_h = \log_2[\lambda_h (\mu_C)]$:

```
In[52]:= yh = Log[(EVh→0)[[3]]] / Log[2]
```

Out[52]= 0.987091

From the scaling relation in Eq. (14) we finally obtain:

```
In[53]:= β = (1 - yh) / yt
```

Out[53]= 0.205158