Nonperturbative square-well approximation to a quantum theory

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The possibility of expressing the solution to a $\phi^{2P}$ quantum field theory as a series in powers of $1/P$ is proposed. Such a series would be nonperturbative in its dependence on the fundamental parameters of the theory such as the mass and the coupling constant. The first term in such a series describes a field in an infinite-dimensional square-well potential. In this paper, the quantum-mechanical Hamiltonian $H = p^2 + q^{2P}$ is studied as a model calculation and the expansion of the energy levels as series in powers of $1/P$ is examined. The method of matched asymptotic expansions to determine the first few terms in the series for all energy levels is used. The results are compared with extensive numerical calculations of the ground-state energy and it is found that the series is extremely accurate: When $P = 2$, the five-term series has a relative error of 6%, when $P = 10$ the relative error is 0.009%, and when $P = 200$ the relative error is $3.4 \times 10^{-9}$%.

I. INTRODUCTION

There have been many attempts to obtain nonperturbative approximations to quantum systems. Such approximations have a clear advantage in that they do not express the content of the theory as series in powers of a physical parameter such as a coupling constant. Thus a nonperturbative approximation may reveal the true dependence of the structure of the theory on the physical parameters. Standard nonperturbative approaches include the $1/N$ expansion in which the field $\phi$ has $N$ components and the Lagrangian has $O(N)$ symmetry, mean-field, and random-phase approximations, and $\epsilon$ expansions. More recently, a nonperturbative expansion called the $\delta$ expansion was proposed in which a $\phi^4$ field theory is approximated by a $(\phi^2)^{1+\delta}$ theory. In this approach, the Green's functions for a $(\phi^2)^{1+\delta}$ field theory are expanded as series in powers of $\delta$ assuming that $\delta \ll 1$. Then $\delta$ is allowed to tend to 1 to obtain the solution to a $\phi^4$ theory. Note that in this approximation scheme one is expanding about a free-field theory because $(\phi^2)^{1+\delta}$ becomes a mass term in the Lagrangian when $\delta = 0$.

In the present paper we propose the possibility of solving a $\phi^4$ quantum theory by expanding the solution to a $\phi^{2P}$ theory as a series in powers of $1/P$ for $P$ large. Note that the $1/P$ series is not an expansion about a free-field theory. Rather, the leading term in the $1/P$ series corresponds to a free field confined to an infinite-dimensional square well. Of course, it is not clear a priori whether such an expansion will be numerically accurate. Furthermore, it is not at all obvious how to calculate such an expansion term by term because when $P = \infty$ the field is completely confined to a finite domain while for $P$ finite there is no such confinement. In other words, the leading term in a $1/P$ series is relatively easy to calculate but subsequent terms may be extremely hard to find.

The purpose of this paper is to illustrate the computation of higher-order terms in the $1/P$ series for a simple quantum-mechanical model. We consider the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + x^{2P} - E(P)\right]\Psi(x) = 0$$

accompanied by the boundary condition

$$\Psi(\pm \infty) = 0.$$  \hspace{1cm} (1.2)

We seek an expansion of the eigenvalue $E(P)$ as a series in powers of $1/P$. To leading order we simply set $P = \infty$. This gives the differential equation

$$\left[-\frac{d^2}{dx^2} - E(\infty)\right]\Psi_0(x) = 0$$

subject to the square-well boundary condition

$$\Psi(\pm 1) = 0.$$  \hspace{1cm} (1.4)

Note that the limit $P \rightarrow \infty$ is a singular limit because, as $P$ becomes infinite, the boundary conditions undergo an abrupt change that reflects the confinement of the wave function to a square-well potential. The eigenvalues of the Schrödinger-equation problem (1.3)–(1.4) are

$$E_n(\infty) = \frac{1}{4} \pi^2 (n + 1)^2, \quad n = 0, 1, 2, \ldots,$$

where $n$ labels the energy level.

The result in (1.5) is the leading term of a series in powers of $1/P$. However, because the perturbation series in powers of $1/P$ is a singular perturbation series, it is not easy to guess the form that such a perturbation series takes. We will see that the eigenvalues have series expansions in powers of $1/P$ and that the coefficient of $P^{-k}$ is a polynomial $Q_k[\ln(2P)]$ of degree $k$ in $\ln(2P)$. Specifically, we will show that

$$E_n(P) = \frac{1}{4} \pi^2 (n + 1)^2 + \sum_{k=1}^{\infty} P^{-k} Q_k[\ln(P)].$$  \hspace{1cm} (1.6)

The series in (1.6) exhibits some remarkable features. First, the quantity $\ln(2P)$ always appears in the combination $\gamma - \ln(2P)$, where $\gamma$ is Euler's constant, $\gamma = - \Gamma'(1) = 0.577215 664 901 \ldots$. Second, we can completely factor out the dependence of the series on the term $\gamma - \ln(2P)$ by writing (1.6) in the form

$$E_n(P) = \frac{1}{4} \pi^2 (n + 1)^2 (2P)^{-2/(2P+1)} \times 
\Gamma\left(\frac{P}{P+1}\right)^2 \left(\sum_{k=0}^{\infty} A_k(n) P^{-k}\right)^2.$$  \hspace{1cm} (1.7)
where the coefficients \( A_k(n) \) are numbers that can be expressed in terms of the Riemann zeta function:

\[
A_0(n) = 1, \\
A_1(n) = 0, \\
A_2(n) = -1 - \frac{1}{2} \zeta(2), \\
A_3(n) = \frac{1}{2} + \zeta(2) - \frac{1}{2} \zeta(3) - \frac{1}{4} \left( \frac{1}{2} \pi^2(n + 1)^2 \right) \zeta(3), \\
A_4(n) = -\frac{1}{3} - \zeta(2) + \frac{1}{2} \zeta(3) + \frac{1}{6} \zeta(4) \\
+ \left[ \frac{1}{2} \pi^2(n + 1)^2 \right] \left( \zeta(3) + \zeta(4) \right).
\] (1.8)

The most efficient way to extract accurate numerical predictions from (1.7) is to convert the series \( \sum_k A_k(n) P^{-k} \) in (1.7) to a Padé approximant. We find that the diagonal series of Padé approximants \( P_0^0, P_1^1, P_2^2, \ldots \) gives the best numerical results, in our case, the vanishing of \( A_1(n) \) prevents us from constructing the first two terms \( P_0^0 \) and \( P_1^1 \) in the diagonal Padé sequence.

To demonstrate the accuracy of the \( 1/P \) expansion in (1.7) we have calculated numerically the ground-state energy \( E_0(P) \) for 60 values of \( P \) in the range \( 1 < P < 3500 \). In Table I, we compare the numerical results for \( E_0(P) \) with the series in (1.6) for some values of \( P \). Observe that the relative error vanishes with increasing \( P \) like \( P^{-3} \). In Figs. 1–4 we plot the relative error in various Padé approximants to the \( 1/P \) series in (1.7) as functions of \( P \). The derivation of the series in (1.6) using the method of matched asymptotic expansions is given in Sec. II.

The accuracy of the \( 1/P \) expansion suggests that one should explore the connection between the \( 1/P \) expansion in this paper and the \( \delta \) expansion in Ref. 1. The connection between these two expansions is easy to establish: Let \( K \) be the highest power of \( 1/P \) in the sum in (1.7). Setting \( P = 1 + \delta \) in (1.7) and expanding the result as a series in powers of \( \delta \) gives for each value of \( K \) an expression for the coefficients of each power of \( \delta \). As \( K \) increases we might expect that the coefficients of each power of \( \delta \) stabilize and become equal to the coefficients in Ref. 1. If this stabilization actually occurs, it must do so in very high order. The coefficient of \( \delta^0 \) is 1 for the ground-state energy. The first five approximations to this number obtained as described above are

\[
1 \quad (K = 0), \\
1 \quad (K = 1), \\
0.834411 \quad (K = 2), \\
1.381265 \quad (K = 3), \\
1.348792 \quad (K = 4),
\] (1.9)

which, if they approach 1 do so rather slowly. From these results we are inclined to believe that the \( \delta \) expansion and the \( 1/P \) expansion do not have a large common region of validity.

**TABLE I.** A comparison of the exact values of \( E_0(P) \) obtained numerically with the predicted values obtained from (1.7) by converting the series \( \sum_k A_k(0) P^{-k} \) to a \( (2,2) \)-Padé for selected values of \( P \). Observe that the relative error decreases like \( P^{-3} \) as \( P \) increases. However, beyond \( P = 200 \) we no longer list the error because the computed eigenvalues are only correct to 11 decimal places.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( E_0(P)_{\text{exact}} )</th>
<th>( E_0(P)_{\text{predicted}} )</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.060 362 090 5</td>
<td>1.121 861 636 6</td>
<td>5.8%</td>
</tr>
<tr>
<td>4</td>
<td>1.225 820 114 1</td>
<td>1.231 689 374 9</td>
<td>0.48%</td>
</tr>
<tr>
<td>5</td>
<td>1.298 843 700 6</td>
<td>1.301 373 512 8</td>
<td>0.19%</td>
</tr>
<tr>
<td>10</td>
<td>1.560 506 342 9</td>
<td>1.560 657 294 6</td>
<td>0.0095%</td>
</tr>
<tr>
<td>20</td>
<td>2.105 213 774 0</td>
<td>2.105 213 866 4</td>
<td>4.3 \times 10^{-9} %</td>
</tr>
<tr>
<td>300</td>
<td>2.337 875 108 9</td>
<td>2.337 875 109 0</td>
<td>3.4 \times 10^{-9} %</td>
</tr>
<tr>
<td>500</td>
<td>2.405 807 979 2</td>
<td>2.405 807 979 2</td>
<td>...</td>
</tr>
<tr>
<td>1500</td>
<td>2.443 094 773 6</td>
<td>2.444 094 773 6</td>
<td>...</td>
</tr>
<tr>
<td>3500</td>
<td>2.455 762 241 3</td>
<td>2.455 762 241 3</td>
<td>...</td>
</tr>
</tbody>
</table>

**FIG. 2.** Same as Fig. 1 for 0.0025 \( \times \) \( 1/(2P) \). \( C < 0.1 \). The straight-line behavior of the curves in the graph imply that the error is of order \( P^{-4} \).
II. DERIVATION OF THE $1/P$ SERIES USING THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

In this section we show how to solve the eigenvalue differential equation in (1.1)-(1.2) for large $P$ using the method of matched asymptotic expansions. Let us summarize the analysis briefly: We decompose the domain $0 < x < \infty$ into three regions. In region 1, where $0 < x < 1$, we can neglect the exponentially small term $x^{2P}$ in the differential equation. Region 2 consists of the neighborhood of $x = 1$. We will specify the size of this region carefully later on. In region 3, where $x > 1$, we neglect the term $E$ because it is small compared to $x^{2P}$, which is exponentially large.

In region 1 the differential equation is trivial because it is a constant-coefficient equation. Even-parity eigenvalues are determined by the requirement that the derivative of the wave function vanish at $x = 0$, and odd-parity eigenvalues are determined by requiring that the wave function vanish at $x = \infty$. The boundary condition that the wave function vanish at $x = \infty$ imposed in region 3. We will solve the differential equation in each of the three regions, impose the boundary conditions at $x = 0$ and $x = \infty$, and require that the wave function satisfy asymptotic matching conditions at the boundaries of regions 1 and 2 and regions 2 and 3. This matching constraint determines the eigenvalues. Specifically, if we carry out an asymptotic match to $k$th order in powers of $1/P$, we determine the eigenvalues $E_n(P)$ correct to $P^{-k}$.

We begin our analysis by introducing a new independent variable:

$$t = E^{-1/(2P)}x, \quad \Psi(x) = y(t).$$  \hspace{1cm} (2.1)

Then (1.1) reads

$$y''(t) + \frac{(\pi^2/4)f^2}{1 - t^{2P}}y(t) = 0,$$  \hspace{1cm} (2.2)

where

$$f = \frac{2}{\pi} E^{1/2 + 1/2P} = f_0 + f_1 \frac{1}{P} + f_2 \frac{1}{P^2} + \cdots.$$  \hspace{1cm} (2.3)

A. Analysis of region 1

Since region 1 consists of those $t$ for which $t^{2P}$ is exponentially small compared with 1, then $y^{(1)}$, the wave function in region 1, satisfies

$$y^{(1)''} + \frac{1}{2} f^2 y^{(1)}(t) = 0.$$  \hspace{1cm} (2.4)

The solution to this equation whose derivative vanishes at the origin is

$$y^{(1)}(t) = A \cos\left[(\pi/2)ft\right],$$  \hspace{1cm} (2.5a)

and the solution that vanishes at the origin is

$$y^{(1)}(t) = A \sin\left[(\pi/2)ft\right],$$  \hspace{1cm} (2.5b)

where $A$ is a constant to be determined by asymptotic matching. Even-parity eigenvalues will come from matching to (2.5a) and odd-parity eigenvalues will come from matching to (2.5b).

B. Analysis of region 3

Region 3 consists of those $t$ for which $t^{2P}$ is exponentially large compared with 1. Thus $y^{(3)}$, the wave function in region 3, satisfies

$$y^{(3)''} + \frac{1}{2} f^2 t^{2P} y^{(3)}(t) = 0.$$  \hspace{1cm} (2.6)

A simple transformation converts this equation to a modified Bessel equation. The solution to (2.6) satisfying the boundary condition $y(\infty) = 0$, is

$$y^{(3)}(t) = C t^{1/2} K_{1/2(P+1)}\left[\left(\pi f/2(P+1)\right) t^{1+P}\right],$$  \hspace{1cm} (2.7)

where $C$ is a constant to be determined by asymptotic matching.
C. Analysis of region 2

Region 2 is the neighborhood of \( t = 1 \). A convenient variable for the treatment of region 2 is

\[ \delta = t - 1. \]  

(2.8)

In terms of \( \delta \) (2.2) becomes

\[ y^{(2)}(\delta) + 4\delta^2 \left[ 1 - e^{2P\ln(1 + \delta)} \right] y^{(2)}(\delta) = 0. \]  

(2.9)

We take \( \delta \ll 1 \) and expand \( \ln(1 + \delta) \) in (2.9):

\[ y^{(2)}(\delta) + 4\delta^2 \left[ 1 - e^{2P\delta} e^{-2P^2\delta^2/3} - e^{-2P^2\delta^2/3} \cdots \right] \times y^{(2)}(\delta) = 0. \]  

(2.10)

Region 2 consists of those \( \delta \) that are small compared with 1. However, in order that overlap regions exist between regions 1 and 2 and regions 2 and 3 where we will perform the asymptotic matching, it is necessary that \( \delta \) must not be too small or else \( y^{(2)} \) will not be exponentially small in region 1 and not be exponentially large in region 3. As we will see, it is sufficient to take

\[ 1/P \ll \delta \ll 1/\sqrt{P} \]  

(2.11a)

to obtain the leading-order (first-order) asymptotic match,

\[ 1/P \ll \delta \ll 1/P^{3/4} \]  

(2.11b)

to obtain the second-order asymptotic match,

\[ 1/P \ll \delta \ll 1/P^{5/6} \]  

(2.11c)

to obtain the third-order asymptotic match, and so on. As we calculate to successively higher orders in powers of \( 1/P \), these asymptotic inequalities provide a self-consistent description of the extent of the matching (overlap) regions. Observe that as the order of perturbation increases, the size of the overlap regions shrinks. This is a well-known and necessary feature of all calculations involving matched asymptotic expansions.\(^3\)

To obtain solutions to (2.10) to any order in \( 1/P \) of the form

\[ y^{(2)} = y_0^{(2)} + \frac{1}{P} y_1^{(2)} + \frac{1}{P^2} y_2^{(2)} + \frac{1}{P^3} y_3^{(2)} + \cdots, \]  

(2.12)

we make the key change of variables

\[ b = \pi f_0/2P, \quad s = be^{\delta}, \]  

(2.13)

where \( f_0 \) is the leading term in the expansion of \( f \) in (2.3) and \( b \) is a small parameter of order \( 1/P \). With a little algebra it is easy to obtain

\[ \exp \left[ 2P \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \delta^k \right] \]

\[ = \left[ \frac{s}{b} \right]^{2n} \sum_{m=0}^{\infty} \frac{2^n \ln(s/b)^m}{m!} \]

\[ \times \left[ \sum_{k=1}^{n} \frac{1}{k} \left[ -\frac{1}{P} \ln \left( \frac{s}{b} \right) \right]^k \left[ \frac{k}{k+1} \right]^m \right] \]

\[ = \left[ \frac{s}{b} \right]^{2n} \sum_{m=0}^{\infty} \frac{1}{P} \beta_n \ln \left( \frac{s}{b} \right), \]

with

\[ \beta_n \left[ \ln \left( \frac{s}{b} \right) \right] = (-1)^n \sum_{m=0}^{\infty} \sigma_{n,m} \ln \left( \frac{s}{b} \right)^{n+m} \]

and

\[ \sigma_{n,m} = 2^n \sum_{a=n}^{n-1} a_1 a_2 \cdots a_n 2^{a_1} 3^{a_2} \cdots (n+1)^{a_n} \sum_{a=0}^{n} a. \]  

(2.14)

and

\[ \left( \frac{f_0}{f_0} \right)^2 = \left[ \sum_{i=0}^{\infty} \frac{1}{P} f_i f_0^2 \right]^2 = \sum_{i=0}^{\infty} \left( \frac{1}{P} \right)^{i} \left[ \sum_{j=0}^{i} \frac{f_{i-j} f_j}{f_0} \right]. \]  

(2.15)

Then (2.10) for all \( n = 0, 1, 2, \ldots \) reads

\[ \frac{d}{ds} \frac{d}{ds} y_0^{(2)}(s) - \sigma_{0,0} s^2 y_0^{(2)}(s) \]

\[ = - \sum_{i=0}^{n-2} \frac{1}{4} \sigma_{i,0} y_0^{(2)}(s) - \sum_{i=0}^{n-2} \frac{1}{2} \sigma_{i,2} y_1^{(2)}(s) \]

\[ + \sum_{k=0}^{n-1} \frac{1}{2} \sigma_{k,0} y_2^{(2)}(s) - \sum_{k=0}^{n-1} \frac{1}{2} \sigma_{k,2} y_1^{(2)}(s) \]

\[ \times \sum_{m=0}^{i} \sigma_{m,m} \ln \left( \frac{s}{b} \right)^{i+m}, \]  

(2.16)

where \( \sigma_{0,0} = 1 \). We will see later that

\[ y_0^{(2)}(s) = 0. \]  

(2.17)

To first order in \( 1/P \), (2.16) is a homogeneous modified Bessel equation of order 0:

\[ s^2 y_1^{(2)}(s) + sy_1^{(2)}(s) - s^2 y_1^{(2)}(s) = 0. \]  

(2.18)

Its general solution is

\[ y_1^{(2)}(s) = K_0(s) + B f_0(s), \]  

(2.19)

where we have exercised our freedom to choose the overall constant in the solution to a homogeneous linear equation by setting the coefficient of the \( K_0 \) function equal to 1. This choice will determine the multiplicative constants of the solutions in regions 1 and 3 when we perform the asymptotic matching.

By virtue of the asymptotic inequalities in (2.11), which now have the form

\[ 1 \ll \ln(s/b) \ll P^{1/2} \]  

(2.20a)

\[ 1 \ll \ln(s/b) \ll P^{1/4} \]  

(2.20b)

\[ 1 \ll \ln(s/b) \ll P^{1/6} \]  

(2.20c)

and so on, we see that the second order in powers of \( 1/P \), (2.16) now reads

\[ s^2 y_2^{(2)}(s) + sy_2^{(2)}(s) - s^2 y_2^{(2)}(s) \]

\[ = s^2 y_1^{(2)}(s) \left[ \frac{2 f_1}{f_0} - 2 \ln \left( \frac{s}{b} \right) \right]. \]  

(2.21)

Unlike the leading-order equation in (2.18), this linear differential equation is inhomogeneous. Using the method of
reduction of order we can write down a formal solution to (2.21):

\[ y^{(2)}_{o}(s) = y^{(2)}_{o}(s) \int_{0}^{b} dx \frac{1}{xK_{o}(x)^{2}} \int_{x}^{\infty} dz K_{o}(z)y^{(1)}_{o}(z) \times \left[ 2 \ln^{2} \left( \frac{x}{b} \right) - \frac{2 f_{1}}{f_{0}} \right] + B_{2}I_{0}(s). \]  

(2.22)

To third order in powers of \(1/P\) (2.16) gives

\[ y^{(3)}_{o}(s) = y^{(2)}_{o}(s) \int_{0}^{\infty} dx \frac{1}{xK_{o}(x)^{2}} \int_{x}^{\infty} dz K_{o}(z)y^{(1)}_{o}(z) \times \left[ \left( \frac{f_{1}}{f_{0}} \right)^{2} - 2 \frac{f_{2}}{f_{0}} + \frac{4 f_{1}}{f_{0}} \ln^{2} \left( \frac{s}{b} \right) - \frac{8}{3} \ln^{2} \left( \frac{s}{b} \right) - \ln^{4} \left( \frac{s}{b} \right) \right] \]

\[ - y^{(2)}_{o}(s) \int_{0}^{\infty} dx \frac{1}{xK_{o}(x)^{2}} \int_{x}^{\infty} dz \frac{1}{z} \frac{1}{f_{0}} K_{o}(z)y^{(1)}_{o}(z) \]

\[ + y^{(2)}_{o}(s) \int_{0}^{\infty} dx \frac{1}{xK_{o}(x)^{2}} \int_{x}^{\infty} dz K_{o}(z)y^{(2)}_{o}(z) \left[ 2 \ln^{2} \left( \frac{x}{b} \right) - \frac{2 f_{1}}{f_{0}} \right] + B_{3}I_{0}(s). \]  

(2.24)

In every case we choose the limits of integration such that the first term vanishes for \(s = b\), the center of region 2, and for \(s = \infty\). This choice will simplify subsequent calculations.

This expansion process can be carried out to any order in powers of \(1/P\). Note that in (2.23)–(2.24), the \([\pi f_{0}(2P)^{2}]\) term in (2.16) contributes for the first time.

D. Matching of regions 2 and 3

The overlap of region 2 and region 3 consists of all \(\delta > 0\) satisfying (2.11a). The positivity of \(\delta\) implies that the arguments of \(y^{(2)}(s)\) and \(y^{(3)}(s)\) are exponentially large. Thus, the solutions must be asymptotically matched for large arguments of the relevant modified Bessel functions. Besides determining the constant \(C\) in (2.7), this match gives the crucial result that to every order in \(1/P\) the exponentially growing contributions to \(y^{(2)}_{o}(s), y^{(2)}_{i}(s), y^{(3)}_{i}(s),\) ... coming from \(I_{o}(s)\) must be eliminated. Thus \(B_{1} = 0, B_{2} = 0,...\).

Since, as we will see, \(y^{(2)}_{o}(s) = 0\), the leading-order term in \(C\) must be of order \(1/P\). In the overlap region we have

\[ y^{(3)}(s) \sim C \sqrt{\frac{\pi}{2s}} e^{-s} \left[ 1 + O \left( \frac{1}{P} \right) + O \left( \frac{1}{s} \right) \right], \]  

(2.25)

and in the same overlap region we have

\[ y^{(2)}(s) \sim \frac{1}{P} y^{(2)}_{o}(s) + O \left( \frac{1}{P^{2}} \right) \]  

(2.25)

\[ - \frac{1}{P} \sqrt{\frac{\pi}{2s}} e^{-s} \left( 1 + O \left( \frac{1}{s} \right) \right) + B_{1} \frac{1}{P} \frac{1}{\sqrt{2\pi s}} \]  

\[ \times e^{s} \left( 1 + O \left( \frac{1}{s} \right) \right) + O \left( \frac{1}{P^{2}} \right). \]  

(2.26)

Aside from the crucial result that \(B_{1} = 0\) for any order \(k\) in \(1/P\), the asymptotic match between solutions in regions 2 and 3 provides no further information. From this knowledge we can now determine the precise form of \(y^{(2)}_{o}(s)\) in (2.22):

\[ y^{(2)}_{o}(s) = K_{o}(s) \left[ a_{2} + \frac{1}{2} \ln \left( \frac{s}{b} \right) \right] + sK_{i}(s) \left[ 1 - \frac{f_{1}}{f_{0}} \ln \left( \frac{s}{b} \right) + \frac{1}{2} \ln^{2} \left( \frac{s}{b} \right) \right], \]  

(2.27)

with

\[ a_{2} = (f_{1}/f_{0} - 1)bK_{1}(b)/K_{0}(b), \]  

(2.27)

where we have evaluated all of the indicated integrals. Similarly, we can simplify the expressions for \(y^{(2)}_{i}(s)\) and \(y^{(3)}_{i}(s)\):
\[ a_3 = \left[ \frac{f_1}{f_0} - 1 \right] a_2 + \frac{1}{2} + \frac{f_2}{f_0} - \frac{f_1}{f_0} \big[ bK_1(b)K_0(b) - \frac{f_1}{f_0} - 1 \big] + b^2 - \frac{1}{4} (\pi^2 f_0 - 1) \frac{I_0(b)}{K_0(b)} \int_0^\infty dx \frac{1}{x} K_0^2(x), \quad (2.29) \]

\[ y_4^{21}(s) = K_0(s) \left[ a_3 + \frac{1}{2} + \frac{f_2}{f_0} - \frac{f_1}{f_0} \right] \ln \left( \frac{s}{b} \right) + \frac{1}{8} \left( 1 - a_2 \right) \ln^{\frac{3}{2}} \left( \frac{s}{b} \right) + \frac{1}{48} \ln^4 \left( \frac{s}{b} \right) \]

\[ + sK_1(s) \left[ \frac{1}{3} - \frac{f_2}{f_0} + \frac{f_1}{f_0} - a_2 + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right) - \frac{1}{2} \left( f_2 - 1 \right) a_2 \right] \ln \left( \frac{s}{b} \right) \]

\[ + \frac{1}{4} \left( \frac{f_1}{f_0} - 1 \right) - \frac{1}{2} f_2 - \frac{1}{2} \left( f_1 - 1 \right) a_2 - a_3 \ln \left( \frac{s}{b} \right) \]

\[ + \frac{1}{8} \left( \frac{f_1}{f_0} - 1 \right) + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right) \ln^2 \left( \frac{s}{b} \right) + \frac{1}{12} \left( f_1 - 1 \right) a_2 \ln \left( \frac{s}{b} \right) \]

\[ + \frac{1}{48} \ln^4 \left( \frac{s}{b} \right) \]

\[ + s^2K_0(s) \left[ \frac{7}{6} - \frac{f_1}{f_0} + \frac{f_1}{f_0} - \frac{f_1}{f_0} - \frac{f_1}{f_0} + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right) a_2 \right] \ln \left( \frac{s}{b} \right) \]

\[ + \left[ - \frac{7}{6} + \frac{1}{4} f_1 \frac{f_1}{f_0} + \frac{1}{6} f_2 \frac{f_1}{f_0} - \frac{1}{2} \left( f_1 - 1 \right) a_2 \right] \ln^2 \left( \frac{s}{b} \right) + \left( \frac{1}{24} - \frac{1}{12} f_2 \right) \ln^3 \left( \frac{s}{b} \right) + \frac{1}{48} \ln^4 \left( \frac{s}{b} \right) \]

\[ + \left( \frac{5}{12} - \frac{1}{4} f_1 \frac{f_1}{f_0} + \frac{1}{2} \left( f_1 - 1 \right) a_2 \right] \ln^3 \left( \frac{s}{b} \right) + \frac{7}{48} \ln^4 \left( \frac{s}{b} \right) - \frac{1}{24} \ln^4 \left( \frac{s}{b} \right) \]

\[ + s^3K_1(s) \left[ - \frac{5}{6} \left( \frac{f_1}{f_0} - 1 \right)^3 - \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right)^2 \ln \left( \frac{s}{b} \right) \right] \]

\[ + \left( \frac{3}{4} - \frac{1}{8} \frac{f_1}{f_0} + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right) \right] \ln \left( \frac{s}{b} \right) + \frac{1}{8} \ln^2 \left( \frac{s}{b} \right) + \frac{1}{48} \ln^3 \left( \frac{s}{b} \right) \]

\[ + \frac{1}{2} \left( \pi^2 f_0 f_1 + \frac{1}{2} - a_2 \right) \left[ I_0(s) \int_0^\infty dx \frac{1}{x} K_0^2(x) + K_0(s) \int_0^\infty dx \frac{1}{x} K_0(x) I_0(x) \right] \]

\[ + \frac{1}{4} \left( \pi^2 f_0 + 1 \right) \left[ I_0(s) \int_0^\infty dx \frac{1}{x} K_0^2(x) \left[ a_2 + \frac{1}{2} \ln \left( \frac{x}{b} \right) \right] \right] \]

\[ + I_0(s) \int_0^\infty dx K_0(x) K_1(x) \left[ 1 - \frac{f_1}{f_0} - \frac{1}{2} \ln \left( \frac{x}{b} \right) \right] + K_0(s) \int_0^\infty dx \frac{1}{x} K_0(x) I_0(x) \left[ a_2 + \frac{1}{2} \ln \left( \frac{x}{b} \right) \right] \]

\[ + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right)^3 b^3 K_0^3(b) - \frac{1}{2} \left( \pi^2 f_0 f_1 + \frac{1}{2} - a_2 \right) I_0(b) K_0(b) \int_0^\infty dx \frac{1}{x} \left( \frac{x}{b} \right)^3 K_0^3(x) \]

\[ + \left( \frac{3}{4} - \frac{1}{2} \frac{f_1}{f_0} + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right) \right] \ln \left( \frac{x}{b} \right) + \frac{1}{2} \ln^2 \left( \frac{x}{b} \right) \right]. \]

\[ (2.30) \]

\[ a_4 = - \left[ 1 - \frac{3}{2} + \frac{f_2}{f_0} - \frac{f_1}{f_0} + 2 \left( 1 - a_2 \right) \right] \frac{bK_1(b)K_0(b)}{K_0(b)} \]

\[ + \frac{5}{6} \left( \frac{f_1}{f_0} - 1 \right)^2 \frac{f_1}{f_0} - \frac{f_2}{f_0} + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right) a_2 \right] b^2 \]

\[ + \frac{1}{6} \left( \frac{f_1}{f_0} - 1 \right)^3 \frac{f_1}{f_0} - \frac{f_2}{f_0} + \frac{1}{2} \left( \frac{f_1}{f_0} - 1 \right) a_2 \right] \frac{bK_1(b)K_0(b)}{K_0(b)} \]

\[ - \frac{1}{4} \left( \pi^2 f_0 + 1 \right) \frac{I_0(b)}{K_0(b)} \left[ I_0(s) \int_0^\infty dx \frac{1}{x} K_0^2(x) \left[ a_2 + \frac{1}{2} \ln \left( \frac{x}{b} \right) \right] \right] \]

\[ + \int_0^\infty dx K_0(x) K_1(x) \left[ 1 - \frac{f_1}{f_0} - \frac{1}{2} \ln \left( \frac{x}{b} \right) \right] + \frac{1}{2} \ln^2 \left( \frac{x}{b} \right) \right]. \]

\[ (2.31) \]
In principle, it is straightforward to obtain \( y_3^{(2)}, \psi_3^{(2)} \) in a similar fashion. Ultimately, however, one encounters difficult integrals of the form

\[
\int_s^\infty dx \frac{1}{x} K_0^2(x) \int_b^\infty dz \frac{1}{z} K_0(z) f_0(z),
\]

which make it difficult to express the expansion in terms of \( b \) or \( s \).

E. Matching of regions 1 and 2

The constant \( A \) in (2.5) is a series in powers of \( 1/P \) beginning with \( (1/P)^0 \). To obtain an asymptotic match here we replace the variable \( t \) in (2.5) by \( 1 + 1/P \ln(s/b) \) and expand the result as a series in powers of \( 1/P \). Using the fact that \( s \) is exponentially small (\( \delta < 0 \)) and \( b \) is of order \( 1/P \), we expand \( y_1^{(2)}(s) = K_0(s) \) and (2.27)–(2.29). Demanding that the expansion of (2.5) agrees order-by-order with the expansion of \( y_3^{(2)}(s) \) in the overlap region gives a sequence of relations for the coefficient \( A \) and the eigenvalue \( E \) in terms of \( f \). To leading order the condition on \( E \)

\[
\cos\left(\frac{\pi}{2} f_0\right) = 0 \quad \text{(2.33a)}
\]

or

\[
\sin\left(\frac{\pi}{2} f_0\right) = 0, \quad \text{(2.33b)}
\]

for the infinite square well, confirming that

\[
y_0^{(2)}(s) = 0, \quad \text{(2.34)}
\]

which gives a discrete spectrum

\[
f_0 = n + 1, \quad n = 0,1,2, \ldots \quad \text{(2.35)}
\]

To higher order, the process of matching establishes equations between coefficients of powers of \( \ln(x/b) \). Solving these equations iteratively gives (1.6). In particular, we find that

\[
Q_1 = \frac{i\pi^2(n + 1)^2(2\nu)}{4},
\]

\[
Q_2 = \frac{i\pi^2(n + 1)^2(2\nu^2 - 2\nu - 2)}{4},
\]

\[
Q_3 = \frac{1}{4} \pi^2(n + 1)^2 \left\{ \frac{4}{3} \nu^3 - 4\nu^2 - 2\nu + 3 + \frac{1}{3} \zeta(3) - \left\lfloor \frac{1}{4} \pi^2(n + 1)^2 \right\rfloor^2 \frac{2}{3} \zeta(3) \right\},
\]

\[
Q_4 = \frac{1}{4} \pi^2(n + 1)^2 \left\{ \frac{2}{3} \nu^3 - 4\nu^2 + 2\nu^2 + \left( 8 + \frac{1}{3} \zeta(3) \right) \nu - \frac{5}{3} - \frac{1}{2} \zeta(3) + \left\lfloor \frac{1}{4} \pi^2(n + 1)^2 \right\rfloor^2 \right\} \times \left[ \frac{4}{3} \nu \zeta(3) + 2\zeta(3) + \frac{1}{2} \zeta(4) \right], \quad \text{(2.36)}
\]

\[
u = \gamma - \ln(2P).
\]

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2We can express a \( \phi^3 \) quantum field theory as a functional integral

\[
Z[J] = \int D\phi \exp\left\{ -\int dx \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + g \phi^3 + J \phi \right] \right\}.
\]

Interpreting this functional integral as a finite product of Riemann integrals on lattice gives

\[
Z[J] = \prod_{n} \left[ \int D\phi_n \exp\left\{ -a^2 \sum_n \left[ \frac{1}{2} (\partial \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 + g \phi_n^3 + J_n \phi_n \right] \right\} \right].
\]

Now, as \( P \to \infty \) we have

\[
Z[J] = \prod_{n} \left[ \int D\phi_n \exp\left\{ -a^2 \sum_n \left[ \frac{1}{2} (\partial \phi_n)^2 + \frac{1}{2} m^2 \phi_n^2 + J_n \phi_n \right] \right\} \right]
\]

and we see that the field at each lattice point \( n \) is restricted to lie in the range \( -1 < \phi_n < 1 \). Thus we have a free field theory confined to an infinite-dimensional square well.