A new perturbative approach to nonlinear partial differential equations

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This paper shows how to solve some nonlinear wave equations as perturbation expansions in powers of a parameter that expresses the degree of nonlinearity. For the case of the Burgers equation \( u_t + uu_x = u_{xx} \), the general nonlinear equation \( u_t + u^2u_x = u_{xx} \) is considered and expanded in powers of \( \delta \). The coefficients of the \( \delta \) series to sixth order in powers of \( \delta \) is determined and Padé summation is used to evaluate the perturbation series for large values of \( \delta \). The numerical results are accurate and the method is very general; it applies to other well-studied partial differential equations such as the Korteweg–de Vries equation, \( u_t + uu_x = u_{xxx} \).

I. INTRODUCTION

In this paper, we will show how to solve a class of nonlinear partial differential equations

\[ u_t + u^2u_x = u_{xx} \tag{1.1} \]

by expanding in powers of \( \delta \). To illustrate our perturbative procedure in the simplest way, we have chosen Gaussian initial conditions. For such initial conditions, we will find that the perturbative analysis can be carried out explicitly to any order in \( \delta \). Note that for \( \delta = 0 \) (1.1) is linear and for \( \delta = 1 \) (1.1) becomes the Burgers equation, a well-known nonlinear wave equation.

Perturbative expansions of the type we use here have been studied intensively in recent years. The simplicity and usefulness of this method is demonstrated in a variety of fields ranging from ordinary differential equations to quantum field theory. The purpose of the present paper is to show that this method can be employed to solve nonlinear partial differential equations with the same ease and success as ordinary differential equations. The central idea of this method is to generalize the original nonlinear problem to a family of problems by introducing a parameter \( \delta \) in the exponent such that for \( \delta = 0 \) the problem becomes linear and solvable whereas for \( \delta = 1 \) the original nonlinear problem is recovered. An expansion in powers of \( \delta \) will then convert the nonlinear problem into an infinite sequence of linear problems which are all formally solvable.

The underlying philosophy of this approach relies on the observation that there is a smooth transition in the behavior of the solution as \( \delta \rightarrow 0 \) for many nonlinear problems. That is, what we perceive as the special case of linearity reveals itself as just another degree on the nonlinearity scale which \( \delta \) measures.

The following treatment will serve as a demonstration that this point of view is also valid for a number of known nonlinear partial differential equations. In Sec. II, we show how to obtain the zeroth- and first-order solutions in \( \delta \) for the generalization of Burgers’ equation in (1.1) using Gaussian initial conditions. We then prove by induction that it is straightforward to calculate any order in perturbation theory explicitly. In Sec. III, we compare the results of a sixth-order calculation in \( \delta \) with the exact solution obtained by numerical integration of the differential equation (1.1). For the chosen times the (3,3)-Padé of the series in \( \delta \) reaches a high degree of accuracy globally in the spatial coordinate. In Sec. IV, we conclude with some remarks concerning the application of the perturbative \( \delta \) expansion to the Korteweg–de Vries equation.

II. PERTURBATIVE \( \delta \)-EXPANSION FOR THE BURGERS EQUATION

We seek a perturbative expansion in powers of \( \delta \) for the solution \( u(x,t) \) of the generalized form of Burgers’ equation in (1.1):

\[ u(x,t) = u^{(0)}(x,t) + \delta u^{(1)}(x,t) + \delta^2 u^{(2)}(x,t) + \cdots. \tag{2.1a} \]

For the initial conditions, we choose

\[ u^{(0)}(x,0) = u(x,0); \quad u^{(n)}(x,0) = 0 \quad \text{for} \quad n > 0. \tag{2.1b} \]

Substituting the formal perturbation series in (2.1a) into (1.1) and expanding the result in terms of \( \delta \) yields

\[ u^{(0)}_t + \delta u^{(1)}_t + \cdots + u^{(n)}_t + \delta u^{(1)}_x + \cdots + \delta^n u^{(0)}_x + \cdots = u^{(0)}_{xx} + \delta u^{(1)}_{xx} + \cdots. \tag{2.2} \]

It is a general feature of this type of expansion that the problem of a nonlinear differential equation decomposes order by order in \( \delta \) into infinitely many inhomogeneous linear problems. For each of these linear problems, the associated homogeneous problem remains the same in any order. The inhomogeneity of the equation of \( n \)th order in \( \delta \) will in general depend on the solution of all previous \( n - 1 \) orders. Of course, for autonomous equations like (1.1), the zeroth-order equation is always homogeneous.

A. Zeroth-order calculation

Keeping only terms to zeroth order in \( \delta \), we get, from

\[ u^{(0)}_t + u^{(0)}_x - u^{(0)}_{xx} = 0. \tag{2.3} \]

The general solution of (2.3) is

\[ u^{(0)}(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} d\xi \phi(\xi) \exp \left\{ -\frac{[(x-t) - \xi]^2}{4t} \right\}, \tag{2.4} \]
where from (2.1b) \( \phi(x) = u^{(o)}(x,0) = u(x,0) \) is the initial condition. For the special case of Gaussian initial conditions, \( \phi(x) = a\sqrt{b/\pi}e^{-bx^2} \),

we obtain from (2.4)

\[ u^{(o)}(x,t) = a\sqrt{4\pi f(t)}^{-1} \exp\left\{- (x-t)^2/[4f(t)]\right\}, \]

where, for convenience, we define

\[ f(t) = t + 1/(4b). \]

Note that \( \lim_{b \to +\infty} \phi(x) = a\delta(x) \).

**B. First-order calculation**

Choosing only terms of first order in \( \delta \), (2.2) yields

\[ u^1 + u^2 - u^3 = - u^0 \ln(u^0). \]

Because this equation is linear, it is easy to obtain a general solution:

\[ u^1(x,t) = \int_0^t \frac{d\tau}{\sqrt{4\pi(f(t) - \tau)}} \int_{-\infty}^{\infty} \exp\left\{ - \frac{(x-t) - (\xi - \tau)^2}{4f(t) - \tau} \right\}, \]

where the inhomogeneity is given by

\[ I^{(1)}(x,t) = - u^2(x,t) \ln(u^0(x,t)). \]

Note that in accordance with (2.1b), \( u^{(1)}(x,0) \equiv 0 \). For the case of the Gaussian initial conditions in (2.5), (2.10) reads explicitly

\[ I^{(1)}(x,t) = - \frac{1}{8f(t)^2} (x-t)^3 \]

\[ + \frac{\ln[4\pi f(t)/a^2]}{4f(t)} (x-t) \]

\[ \times \left\{ a/\sqrt{4\pi f(t)} \right\} \exp\left\{- (x-t)^2/[4f(t)]\right\}. \]

If we insert (2.11) into (2.9) and make the change of variable

\[ \xi - \xi' + \tau + [f(t)/f(t)](x-t), \]

we obtain a simple integral of a Gaussian multiplying a polynomial in \( \xi' \). It is remarkable that the residual exponent after the shift in (2.12) does not depend on \( \tau \), but rather reproduces \( u^{(0)}(x,t) \). Thus, after evaluating the \( \xi' \) integral, we are left with trivial integrals of the form

\[ \int_{-\infty}^{\infty} \left\{ \ln\left[4\pi f(t)/a^2\right] \right\}^k \frac{1}{\sqrt{\pi}} e^{-\tau f(t)\tau} d\tau, \]

with \( k, l, m \in \mathbb{Z} \) such that the integrals converge. The final result is

\[ u^{(1)}(x,t) = u^{(0)}(x,t) \left\{ - \frac{1}{16f(t)} + \frac{1}{256f(t)^2 b^2} \right\} (x-t)^3 \]

\[ + \left\{ - \frac{1}{8} - \frac{1}{4} \ln\left(4\pi f(t)/a^2\right) + \frac{1}{8f(t)b} \right\} (x-t)^2 \]

\[ + \frac{1}{16f(t)b} \ln\left(\frac{\pi}{a^2 b^2}\right) - \frac{3}{128f(t)^2 b^2} \right\} (x-t). \]

\[ \text{(2.13)} \]

**C. Calculation in nth order**

It is quite clear now that to nth order in \( \delta \) we will have to solve the linear, inhomogeneous partial differential equation

\[ u^{(n)} + u^{(n)}_x - u^{(n)}_{xxx} = I^{(n)}(x,t). \]

To obtain \( I^{(n)}(x,t) \) we observe that we can rewrite (1.1) as

\[ u_x + u_x - u_{xx} = - u_x(u^2 - 1). \]

Next we expand \( u^2 - 1 \),

\[ u^2 - 1 = \sum_{n=1}^{\infty} \delta^n [\ln(u)]^n, \]

and substitute (2.1a) for \( u \). Reexpanding this series in powers of \( \delta \) gives

\[ u^{(n)} - 1 = \sum_{n=1}^{\infty} \delta^n [\ln(u)]^n \]

\[ \times \sum_{m=0}^{\infty} \sum_{l=0}^{m} \sum_{k=0}^{l} \left( \frac{1}{a^2} \right)^{m-l} \frac{1}{b^2} \frac{1}{k!} \frac{1}{l!} \frac{1}{m!} \frac{1}{n!} \]

\[ \times \sum_{\xi_i \neq \xi_j} \sum_{\eta_{i,j}} \sum_{i,j,k} \frac{F_i[u]b_i^2 \cdots F_k[u]b_k^2}{b_i! \cdots b_k!} \]

\[ \text{(2.16a)} \]

Next, we multiply (2.16a) by \( u_x \), where \( u \) is given by (2.1a). Expanding in powers of \( \delta \) gives the right side of (2.15):

\[ - u_x(u^2 - 1) = \sum_{n=1}^{\infty} \delta^n I^{(n)}(x,t), \]

\[ \text{(2.17a)} \]

with

\[ I^{(n)}(x,t) = - \sum_{m=1}^{\infty} u^{(m)}(x,t) \sum_{l=0}^{\infty} \sum_{k=0}^{l} \left( \frac{1}{a^2} \right)^{m-l} \frac{1}{b^2} \frac{1}{k!} \frac{1}{l!} \frac{1}{m!} \frac{1}{n!} \]

\[ \times \sum_{\xi_i \neq \xi_j} \sum_{\eta_{i,j}} \sum_{i,j,k} \frac{F_i[u]b_i^2 \cdots F_k[u]b_k^2}{b_i! \cdots b_k!} \]

\[ \text{(2.17b)} \]

For the case of Gaussian initial conditions (2.5), we observe now that if, for all \( i < n \), \( u^{(i)}(x,t) \) has the form

\[ u^{(i)}(x,t) = \sum_{k=0}^{\infty} a_k^{(i)}(t) (x-t)^k \]

\[ \text{(2.18)} \]

then

\[ I^{(n)}(x,t) = \sum_{k=0}^{\infty} H_k^{(n)}(x,t) (x-t)^k u^{(0)}(x,t). \]

\[ \text{(2.19)} \]

To prove this observation, we merely need to point out that, if (2.18) is true, then all (\( u^{(i)}/u^{(0)} \)) in (2.16b) and consequently \( F_i[u] \) are polynomials in \( (x-t) \) with coefficients depending on \( t \). Since

\[ \ln(u^{(0)}) = - \frac{1}{4f(t)} (x-t)^2 - \frac{1}{2} \left( \frac{4\pi f(t)}{a^2} \right) \]

and
\[ u_x^{(m)} = \left( \frac{u^{(m)}}{u^{(0)}} \right)_x + \left( \frac{u^{(m)}}{u^{(0)}} \right) u^{(0)}, \tag{2.21a} \]

with

\[ \frac{u^{(0)}}{u^{(0)}} = - \frac{1}{2f(t)} \left( x - t \right) \tag{2.21b} \]

are also polynomials having this form, then (2.19) follows from (2.17b). Furthermore, by construction we can prove that, if (2.19) holds for \( I^{(n)} \), then \( u^{(n)} \) will be of the form (2.18). Therefore, the generalization of (2.9) from \( n = 1 \) to arbitrary \( n > 0 \) reads

\[ u^{(n)}(x,t) = \int_0^t \frac{d\tau}{\sqrt{4\pi f(t) - f(\tau)}} \]
\[ \times \int_{-\infty}^\infty d\xi \left\{ \sum_{k=0}^{n} H_k^{(n)}(\xi)(\tau - \xi)^k u^{(0)}(\xi,\tau) \right\} \]
\[ \times \exp \left\{ - \frac{[x - t - (\xi - \tau)]^2}{4f(t) - f(\tau)} \right\}, \tag{2.22} \]

or

\[ u^{(n)}(x,t) = \frac{a}{\sqrt{4\pi f(t)}} \left( \sum_{k=0}^n \frac{k!}{(n-k)!} \right) \]
\[ \times \int_0^t \frac{d\tau H_k^{(n)}(\tau)}{\sqrt{4\pi f(t) - f(\tau)}} \]
\[ \times \int_{-\infty}^\infty d\xi (\xi - \tau)^k \exp \left\{ - \frac{\xi^2}{4f(t) - f(\tau)} \right\}, \tag{2.23} \]

Note that \( u^{(n)}(x,0) \equiv 0 \) in accordance with (2.1b).

Again, shifting \( \xi \to \xi' + \tau + [f(\tau)/f(t)](x - t) \) as in (2.12), we obtain

\[ u^{(n)}(x,t) = u^{(0)}(x,t) \left( \sum_{k=0}^n \frac{k!}{(n-k)!} \right) \]
\[ \times \int_0^t \frac{d\tau H_k^{(n)}(\tau)}{\sqrt{4\pi f(t)}} \int_{-\infty}^\infty d\xi (\xi - \tau)^k \exp \left\{ - \frac{\xi^2}{4f(t) - f(\tau)} \right\}. \tag{2.24} \]

Finally, we use

\[ \int_{-\infty}^\infty d\xi (\xi - \tau)^r \exp \left\{ - \frac{\xi^2}{\alpha} \right\} = \begin{cases} \sqrt{\pi} r^{r/2} \Gamma \left( \frac{r+1}{2} \right), & \text{if } r \text{ even} \\ 0, & \text{if } r \text{ odd} \end{cases} \tag{2.25} \]

and rearrange the summation to get

\[ u^{(n)}(x,t) = u^{(0)}(x,t) \left( \sum_{k=0}^n a_k^{(n)}(t) (x - t)^k \right), \tag{2.26} \]

with

\[ a_k^{(n)}(t) = \frac{1}{k! f(t)^k} \sum_{i=0}^{[n-k/2]} \sum_{j=0}^{k/2} \frac{(2i+k)!}{i! j! (i+j)!} \]
\[ \times (-1)^i i^i j^j \int_0^t d\tau H_{2i+k}^{(n)}(\tau) f(\tau)^{2i-j}. \tag{2.27} \]

This completes the proof.

III. NUMERICAL RESULTS

Before we proceed with the presentation of some numerical results, it is instructive to conduct a dimensional analysis of (1.1). Related to this is the subject of minimal sensitivity that we address later. The subsequent numerical results show that with such a high-order approximation in the \( \delta \) expansion a high degree of accuracy can be reached at \( \delta = 1 \), even without imposing the condition of minimal sensitivity.

A. Scaling properties of the Burgers equation

To provide (1.1) with the physically correct dimensions, we note that since \( u \) has dimensions of velocity, we must introduce a velocity \( M \) and a diffusion coefficient \( \nu \) such that

\[ u_x + M(u/M)u_{xx} = \nu u_{xx}. \tag{3.1} \]

By making the scale transformations

\[ x = (\nu/M)x, \quad t = (\nu/M^2)t, \quad u = MU, \tag{3.2} \]

we obtain dimensionless quantities \( X, T, \) and \( U \) and, with those, we reproduce (1.1). Note also that the initial conditions need to be rescaled accordingly. For instance, for the Gaussian initial conditions in (2.5), we need to scale

\[ a = \sqrt{A}, \quad b = (M^2/\alpha)B, \quad \phi = MF. \tag{3.3} \]

We then obtain

\[ \Phi(X) = F(\sqrt{B/\pi})e^{-\alpha X^2}. \tag{3.4} \]

where

\[ \Phi(X) \equiv U(X,0). \tag{3.5} \]

Because of (3.3), for any choice of \( a \) and \( b \), the perturbative solution of (1.1) will always be implicitly dependent on \( M \) and \( \nu \). This dependence can be made explicit by using the transformations (3.2).

B. Minimal sensitivity

Note that at \( \delta = 1 \), (3.1) is independent of \( M \). However, the velocity parameter \( M \) appears in every finite order of perturbation theory. Apparently then, as the order of perturbation theory increases, the results should become less and less sensitive to the value of \( M \) when \( \delta = 1 \). Thus, if we are interested in the solution to (1.1) at \( \delta = 1 \), in every order in perturbation theory, it seems reasonable to choose the value of \( M \) to be the point at which the perturbative solution is least sensitive to changes in \( M \). Thus, one could compute the derivative with respect to \( M \) of the perturbation expansion in \( n \)th order and set the result equal to 0. This gives an algebraic equation which can, in principle, be solved for \( M \). The resulting solution for \( M \) can then be substituted back into the nth-
FIG. 1. A comparison of the exact solution to the generalized Burgers equation in (1.1) for the Gaussian initial conditions in (2.5) with $a = 5$ and $b = 1$ and the first six Padé approximants in the diagonal sequence, $P_0^s$, $P_1^s$, $P_2^s$, $P_3^s$, $P_4^s$, $P_5^s$, constructed from the $\delta$ expansion at $\delta = 1$. Both graphs in each of the upper diagrams display $u(x,t)$ at $t = 1$. Below each comparison is a plot of the relative error $|e_{rel}|$ between the exact solution and Padé approximation. Observe that except in the vicinity of the poles of the Padé approximants, the numerical accuracy is good. Poles appear as upward spikes in the relative error plots. Note that as the order of the $\delta$ expansion increases, the widths of the neighborhoods of the poles in which the Padé approximation is inaccurate shrink rapidly to zero. The downward spikes on the error plots are points where the Padé approximation happens to cross the exact solution.
order perturbation series. It is interesting that the value of $M$ obtained in this way is actually not a constant as we have assumed all along but rather has a dependence on both space and time. Thus, the principle of minimal sensitivity elevates what was initially a constant to a space-time dependent classical field. Some initial experimentation shows that the accuracy of a given perturbation calculation increases dramati-
cally when the principle of minimal sensitivity is used. However, the resulting equations are complicated and, furthermore, our high-order perturbation calculations are so accurate that we see no reason at this point to discuss the principle of minimal sensitivity further in this paper. Our approach here is simply to take $M = 1$ and proceed with the calculation.
C. Numerical results

The results of the $\delta$-expansion calculation for the Gaussian initial conditions in (2.5) with parameters $a = 5$ and $b = 1$ are given in Figs. 1–4. On these figures, we consider four values of $\delta$: $\delta = 1/2$, $1$ (Burgers' equation), $2$, and $4$. (Only the case $\delta = 1$ is analytically solvable by means of the Hopf Cole transformation.) On each of these figures, we compare the exact solution at $t = 1$ found numerically with the first six elements of the diagonal Padé sequence, $P_0^1$, $P_1^1$, $P_2^2$, $P_3^3$, and $P_4^3$, constructed from the $\delta$ expansion. We have used a Padé summation here to improve the rate of convergence of the $\delta$ perturbation series. We do not know what the radius of convergence of this series is. However, our

![Diagram](image)
experience\(^1\) has been that the accuracy of the \(\delta\) series is significantly enhanced by Padé summation. We also plot the relative error between the exact solution at \(t = 1\) and the Padé approximation. Observe that, as one might expect, the accuracy of the Padé decreases slightly as \(\delta\) increases away from 0. However, except in the vicinity of poles (zeros of the denominator) the Padé approximations are good.

IV. PERTURBATIVE \(\delta\) EXPANSION OF THE KORTEWEG-DE VRIES EQUATION

To demonstrate the generality of the perturbative \(\delta\) expansion, we use it in this section to obtain at least formally the solution to a two-parameter class of nonlinear partial differential wave equations:

![Graphs showing the solution to the Korteweg-de Vries equation for different Padé approximations.](image)

FIG. 4. Same as in Fig. 1 except that \(\delta = 4\).
A. Perturbative δ expansion for (4.1)

If we substitute the series (2.1a) into (4.1), the problem in (4.1) decomposes to

\[ u_i^{(0)} + u_x^{(0)} - \partial_\tau^m u^{(0)} = 0, \quad m = 1, 2, 3, \ldots \] \hspace{1cm} (4.2a)

\[ u_i^{(n)} + u_x^{(n)} - \partial_\tau^m u^{(n)} = I^{(n)}(x, \tau), \quad m = 1, 2, 3, \ldots \] \hspace{1cm} (4.2b)

with the appropriate initial conditions given in (2.1b). We point out that \( I^{(n)}(x, \tau) \) will in general be some complicated function depending on all previous \( u_i^{(0)}(x, \tau) \), \( i < n \). The solution of (4.2a) for general initial conditions \( \phi(x) = u(x, 0) = u_i^{(0)}(x, 0) \) is

\[ u_i^{(0)}(x, \tau) = \int_{-\infty}^{\tau} \frac{dp}{\sqrt{4\pi}} \phi(\zeta) \int_{-\infty}^{\infty} \frac{dp}{\sqrt{4\pi}} \exp\left\{ - \frac{1}{4\pi} \right\} \times \exp\left\{ -ipm^m + ip[\zeta - (x - t)] \right\}, \quad m = 1, 2, 3, \ldots \] \hspace{1cm} (4.3)

The solution of (4.2b) with \( u_i^{(n)}(x, \tau) = 0 \) for \( n > 0 \) is

\[ u_i^{(n)}(x, \tau) = \int_{-\infty}^{\tau} d\tau \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} I^{(n)}(\zeta, \tau) \times \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \exp\left\{ - \frac{1}{2\pi} \right\} \exp\left\{ -ipm^m + ip[\zeta - (x - t)] \right\}, \quad m = 1, 2, 3, \ldots \] \hspace{1cm} (4.4)

An interesting special case is obtained by imposing the Gaussian initial conditions in (2.5). With these initial conditions, we get for (4.3)

\[ u_i^{(0)}(x, \tau) = \pm \left\{ \frac{b}{\pi} \right\} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \exp\left\{ - \sqrt{\frac{b}{\pi}} \right\} \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \exp\left\{ - \frac{1}{2\pi} \right\} \exp\left\{ -ipm^m + ip[\zeta - (x - t)] \right\} \times \exp\left\{ -ipm^m + ip[\zeta - (x - t)] \right\}, \quad m = 1, 2, 3, \ldots \] \hspace{1cm} (4.5)

Evaluating the Gaussian integral gives

\[ u_i^{(0)}(x, \tau) = \frac{a}{2\pi} \int_{-\infty}^{\infty} dp \exp\left\{ - \frac{1}{4\pi} \right\} \times \exp\left\{ -ipm^m + ip[\zeta - (x - t)] \right\}, \quad m = 1, 2, 3, \ldots \] \hspace{1cm} (4.6)

B. Korteweg–de Vries equation

For the particular case of the Korteweg–de Vries equation, we choose \( m = 3 \) in (4.6) and obtain

\[ u_i^{(0)}(x, \tau) = \frac{a}{2\pi} \int_{-\infty}^{\infty} dp \exp\left\{ - \frac{1}{4\pi} \right\} \times \exp\left\{ -ipm^m + ip[\zeta - (x - t)] \right\}, \quad m = 1, 2, 3, \ldots \] \hspace{1cm} (4.7)

A shift \( \zeta - \tau' - i/\left(12bt\right) \) eliminates the quadratic term in the exponent with the result that

\[ u_i^{(0)}(x, \tau) = \frac{a}{\pi} \exp\left\{ - \frac{1}{864b 12t^2} \right\} e^{- (x - t)/12bt} \times \cos\left\{ \frac{1}{48b} t^3 - (x - t) \right\} \] \hspace{1cm} (4.8)

We easily identify the above integral as a representation of the Airy function \( \text{Ai} \). Thus,

\[ u_i^{(0)}(x, \tau) = \frac{a}{(3t)^{1/3}} \exp\left\{ - \frac{1}{864b 12t^2} \right\} e^{- (x - t)/12bt} \times \text{Ai}\left\{ \frac{1}{48b} t^3 - (x - t) \right\} \] \hspace{1cm} (4.9)

is the leading term in the solution of the Korteweg–de Vries equation for Gaussian initial conditions in the perturbative \( \delta \) expansion.

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