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A conjecture on the maximum cut and bisection width in random regular graphs

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Abstract. The asymptotic properties of random regular graphs are objects of extensive study in mathematics and physics. In this paper we argue, using the theory of spin glasses in physics, that in random regular graphs the maximum cut size asymptotically equals the number of edges in the graph minus the minimum bisection size. Maximum cut and minimal bisection are two famous NP-complete problems with no known general relation between them; hence our conjecture is a surprising property for random regular graphs. We further support the conjecture with numerical simulations. A rigorous proof of this relation is an obvious challenge.

Keywords: cavity and replica method, spin glasses (theory), random graphs, networks

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Contents

1. Introduction

The maximum cut and the minimal bisection problems are famous in graph theory. Given a graph, i.e. a set of nodes V and a set of edges E, the goal in the maximum cut problem is to split the set of nodes into two groups in such a way that the number of edges connecting the two groups is the largest possible. In the minimal bisection problem the goal is to split the set of nodes into two equally sized groups in such a way that the number of edges between the two groups is the smallest possible. Minimal bisection is also known under the name of graph bi-partitioning. Both of these problems are recognized as NP-complete [1], and both have a large number of applications in computer science and engineering. Some intensively studied applications of the graph partitioning problem are circuit design [2] and data clustering and load balancing in parallel computing [3]. For applications of the 'max-cut' problem see, e.g., the survey article in [4].

Random r-regular graphs are randomly chosen from all those graphs having N nodes and the degree of each node fixed to r. Determining the asymptotic size of their max-cut or their 'min-bisection' (bisection width) is a classical problem in random graph theory; see [5]–[10] for the best known lower and upper bounds. However, as far as we know, no explicit relation between the max-cut size and the bisection width is known in the graph theoretical literature. An exception is provided in [10], where the same approximative algorithm is used to provide an upper bound for the bisection width and a lower bound for the max-cut.

The main purpose of this paper is to conjecture that in random regular graphs the size of the max-cut is asymptotically equal to the number of edges minus the size of the min-bisection. Our conjecture states that in the large N limit,

$$|\mathrm{MC}| = |E| - |\mathrm{BW}| + \mathrm{o}(|\mathrm{BW}|), \tag{1}$$

where $|E|$ is the total number of edges. This surprising result arises from the cavity method, a powerful, albeit non-rigorous, method developed in physics to treat spin glasses [11, 12].

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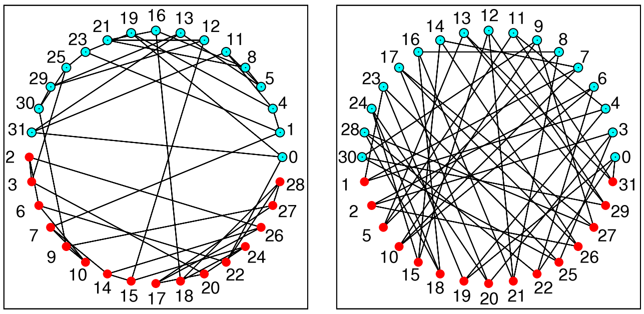

Figure 1. Two different drawings of the same randomly generated 3-regular graph with $N = 32$ nodes. Left: example of a minimal bisection of the graph; only six edges are present between the group of blue (up) and red (down) nodes. Right: example of a maximum cut; only five edges are present between two nodes of the same color.

In figure 1, we present two different drawings of the same 3-regular graph with $N = 32$ nodes. The left side is a minimal bisection of size $|\mathrm{BW}| = 6$; the right side shows a maximum cut of size $|\mathrm{MC}| = rN/2 - 5$. It illustrates that equation (1) is a highly non-intuitive result, since on a given graph there is no straightforward relation between the set of edges in the maximum cut and the minimal bisection. As we will show in the rest of this paper, hints as regards this conjecture already appeared in various forms in the spin glass literature. Our goal is to collect arguments for its justification, state them in a language that does not require knowledge of the replica or cavity computations, provide evidence from precise numerical simulations, and most importantly, clarify the conditions under which this conjecture holds and discuss its generalizations.

2. Statistical physics formulation of the problems

In statistical physics, the max-cut and bi-partitioning can be formulated in terms of finding the ground state of an Ising model on random r-regular graphs. For any graph, the general Ising model Hamiltonian reads

$$\mathcal{H} = -\sum_{(ij)\in E} J_{ij} S_i S_j, \tag{2}$$

where the sum extends over all edges in the graph, J_{ij} is the interaction strength, and $S_i \in \{-1, +1\}$ are the Ising spin variables. The max-cut problem is cast as a ground state of the antiferromagnetic Ising model, i.e. minimization of (2) with $J_{ij} = -1$ for all $(ij) \in E$ with respect to the values of the spins $\{S_i\}$. The min-bisection, or graph bi-partitioning, is a ground state of the ferromagnetic Ising model with magnetization fixed to zero, i.e. minimization of (2) with $J_{ij} = 1$ for all $(ij) \in E$ subject to a constraint $\sum_i S_i = 0$. Let $E_{\mathrm{GS}}(\{J_{ij}\})$ be the energy of the corresponding ground state; then the size of the bisection width and the max-cut are

$$|\mathrm{BW}| = \frac{|E| + E_{\mathrm{GS}}(\{J_{ij}\})}{2}, \qquad |\mathrm{MC}| = \frac{|E| - E_{\mathrm{GS}}(\{J_{ij}\})}{2}, \tag{3}$$

where $|E|$ is again the total number of edges.

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One can interpolate between the max-cut and min-bisection problems by taking the interactions J_{ij} uniformly at random from a bimodal distribution

$$P(J_{ij}) = \rho\delta(J_{ij}+1) + (1-\rho)\delta(J_{ij}-1) \tag{4}$$

and by fixing the magnetization to zero when needed. The disorder in the interactions induces frustration on any kind of loopy lattice, in which case the Hamiltonian (2) then provides a model for a spin glass [11]. For sparse random regular graphs the conjecture discussed here can be generalized as: the ground state energy of (2) is asymptotically independent of ρ, i.e. in the large N limit it is

$$E_{\mathrm{GS}}(\{J_{ij}\},\rho) = Ne_{\mathrm{GS}} + \mathrm{o}(N), \qquad (N\to\infty, 0\le\rho\le 1). \tag{5}$$

3. Previous results

In the statistical physics of disordered systems, the replica or cavity method [11, 12] and a replica symmetry breaking scheme [13] are used to compute the exact ground state of Hamiltonian (2) at zero magnetization. Unfortunately, these techniques are not rigorous, although in many models their results have been proven; see e.g. [14, 15].

Using the replica method, Fu and Anderson [16] computed the ground state energy of a graph bi-partitioning ($\rho = 0$) on dense random graphs, i.e. when the degree $r = pN$ ($0 < p \le 1$) is a constant independent of the graph size N. Their result reads

$$E_{\mathrm{GS}}^{\mathrm{dense}} = U_{\mathrm{SK}} N^{3/2}\sqrt{p(1-p)} + \mathrm{o}(N^{3/2}), \tag{6}$$

where U_{SK} is the ground state energy density of the Sherrington–Kirkpatrick model [17] computed using the Parisi formula [13], a numerical evaluation giving $U_{\mathrm{SK}} = -0.763\,219\ldots$. They obtained this result by realizing that on the dense graphs the replica equations for the Hamiltonian (2) at zero magnetization are basically identical to the replica equations for the Sherrington–Kirkpatrick model. Moreover, the minimal bisection of a graph G plus the maximum cut of the complement of G (i.e., the graph composed of edges that are not present in G) equals $(N/2)^2$. Using this identity plus equations (3) and (6), we obtain that the bisection width is the number of edges minus the size of the max-cut $|\mathrm{BW}| = pN^2/2 - |\mathrm{MC}| + \mathrm{o}(N^{3/2})$.

Similarly, using results on the ground state energy of the spin glass, $\rho = 1/2$, references [18]–[21] computed the bisection width on sparse random regular graphs. In the sparse case, the size of the bisection width is linear in the size of the system and one therefore obtains $|\mathrm{BW}| = |E| - |\mathrm{MC}| + \mathrm{o}(N)$. This relation can be proven rigorously on sparse random graphs in the limit of large degree up to the first two orders in the inverse degree; in particular, $|\mathrm{BW}| = rN/2 + U_{\mathrm{SK}}N\sqrt{r} + \mathrm{o}(\sqrt{r}) + \mathrm{o}(N)$ [22].

However, the existing literature never discusses for what ensembles of random graphs the above results hold. A counter-example is provided by the Erdős–Rényi random graphs, where every edge is present with probability $\alpha/(N-1)$. On the Erdős–Rényi random graphs the spin glass model, $\rho = 1/2$, and the max-cut, $\rho = 1$, have a positive ground state energy above the percolation threshold, $\alpha > 1$. In contrast, the bisection width, $\rho = 0$, of an Erdős–Rényi graph is positive only above $\alpha = 2\ln 2$, at which point the giant component reaches size $N/2$. Thus, we need to discuss the theoretical arguments for sparse random graphs and specify the conditions under which the max-cut and bisection width are related.

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Note also that on dense graphs, although conjecture (1) holds, the ground state energy of (2) is not independent of ρ. Hence, generalization (5) holds only on sparse graphs. For example, for $p = 1$ and 0 the ground state energy is zero, whereas for $\rho = 1/2$ the ground state energy is $U_{\rm SK} N^{3/2}$.

For completeness, let us note that the model (2) on random graphs without the condition on zero magnetization was studied in [23]. It was found that there is an r-dependent critical value of $0 < \rho_{\rm c}(r) < 1/2$ such that for $\rho > \rho_{\rm c}(r)$ the models with zero magnetization and non-fixed magnetization are asymptotically equivalent, and for $\rho < \rho_{\rm c}(r)$ the two are different as the second develops a non-zero magnetization. From the one-step replica symmetry breaking solution, reference [23] found, e.g., $\rho_{\rm c}(3) = 0.142$ and $\rho_{\rm c}(r) \to 1/2$ for $r \to \infty$.

4. Theoretical arguments

We will argue that on sparse random graphs the ground state of (2) at zero magnetization does not depend on the fraction of antiferromagnetic bonds ρ. The main part of the argument is based on the fact that sparse random graphs (with finite mean of the degree distribution) are locally tree like, i.e. the length of the shortest cycle passing through a random node diverges when $N \to \infty$. On a tree, all dependence on ρ can be 'pushed' into the boundary conditions by using recursively from the root the gauge transformations $J_{ij} \to \sigma_i J_{ij} \sigma_j$ and $s_i \to \sigma_i s_i$, where $\sigma_i \in \{\pm 1\}$ are chosen in such a way as to yield, say, $J_{ij} = -1$ for all (ij).[3]

Note that this gauge transformation always conserves the Hamiltonian (2). So we 'only' need to discuss the dependence on the boundary conditions. There are a set of properties on the boundary conditions that do influence behavior in the bulk of the tree; let us call these the *relevant* properties. To make the connection between the systems on trees and on the random graph, we need to consider boundary conditions on the tree having the same *relevant* properties as a configuration taken uniformly at random from the zero-temperature Boltzmann measure associated with (2) on the random graph.

Moreover, in order to argue in favor of our conjecture (5), we need the *relevant* properties of the boundary conditions to stay unchanged after the gauge transformation. A particular *relevant* property is the total magnetization on the boundary conditions. Note that gauge flipping of any finite fraction of random bonds in the tree causes a random half of the spins on the boundary conditions to flip. Hence, only the zero value of magnetization can be treated in this way. This is a first important limitation of the conjecture (5)—the independence of ρ of the ground state of (2) at zero magnetization does not generalize to non-zero values.

We said that all the *relevant* properties, not only the magnetization, of the boundary conditions need to be conserved by the gauge transform. This leads to an impasse in the mathematical rigor of our discussion and we have to resort to non-rigorous arguments implicit in the cavity method. For the cavity method approach, Mézard and Parisi [12] argue that the space of configurations and boundary conditions can be split into states. Every state has an associated set of boundary conditions given in such a way that within

[3] Any other pre-defined configuration of J_{ij}s would do as well.

each there is no dependence of the bulk properties on the precise boundary conditions corresponding to the state. This notion is familiar from the Ising ferromagnet in the low temperature phase, where there are two such states. Mézard and Parisi [12] treat the case where the number of states grows exponentially with the size of the system; this is called replica symmetry breaking. If there is no dependence of the bulk on the boundary conditions, then there are no *relevant* properties, and an empty set is certainly conserved by the above gauge transformation. Thus, to finish our argument we 'only' need to show that the solution of the cavity equations (that describe the splitting into states) is independent of ρ or, in other words, conserved by the gauge transformation.

The cavity equations are written in terms of local magnetic fields and their distributions over the graph edges (and over the different states, if the corresponding problem is glassy)[4]. It follows from the cavity equations that, if there is a global symmetry between positive and negative fields, then the system has to have zero magnetization. However, the opposite is not true: the requirement of zero magnetization does now imply the distribution of fields to be symmetric around zero. The inhomogeneity in the graph degree may lead to zero magnetization without overall plus–minus symmetry, as can be illustrated again by the example of Erdős–Rényi graphs above the percolation threshold; see [24]. There, at $\rho = 0$ the denser parts of the graph are more likely to have positive fields, and the sparser parts have excessive negative fields (or vice versa). Again, such a non-trivial symmetry breaking is not conserved by the gauge transform and, hence, on Erdős–Rényi graphs the ground state of (2) may be (and in fact is) dependent on ρ. Other ensembles of non-regular random graphs may also have this degree fluctuation driven symmetry breaking and hence no reason for validity of (5), in particular for low values of ρ where the equilibrium value of magnetization is not zero[5].

Random regular graphs, on the other hand, have no inhomogeneity in degree and they locally look the same from any node in the graph. Moreover, for $\rho = 0$ and 1 there is no inhomogeneity in the interactions J_{ij} either; hence, the cavity fields (or their distributions over states) have to be the same on every edge. In such a case, the only way to obtain zero magnetization is to have cavity field distributions symmetric around zero. Hence one obtains the same cavity equations for both the graph bisection ($\rho = 0$) and the max-cut ($\rho = 1$) problems. For the remaining values of $0 < \rho < 1$, the neighborhood of every node is different in terms of the set of interactions J_{ij}. However, this difference can be pushed to the boundary conditions via the gauge transformation. And from the independence of boundary conditions in every state it follows that the distribution of fields is the same on every edge.

In conclusion, the solutions of the cavity equations for the ground state of (2) at zero magnetization are the same for every $0 \leq \rho \leq 1$; this is true on any level of replica symmetry breaking and, consequently, the ground state energy of (2) is independent of ρ as long as the graph of interactions is regular.

[4] In computer science the local magnetic fields are known as the beliefs in the belief propagation algorithm, and different states correspond to different belief propagation fixed points.

[5] On the other hand the independence of ρ may be valid for ρ above some critical value. For example, the spin glass $\rho = 1/2$ is equivalent to the antiferromagnet $\rho = 1$ for other ensembles of sparse random graphs, where the bisection $\rho = 0$ is not.

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5. Numerical evidence

We use the extremal optimization (EO) heuristics [25, 29] to find ground states of (2) at zero magnetization for different values of ρ. There has been previous use of the EO heuristics for finding ground states on random graphs for graph bi-partitioning ($\rho = 0$) [26]–[28], [30], and for spin glasses ($\rho = 1/2$) [31, 33]. Thus, this approach is perfectly suited to approximating ground states over the entire range of $0 \leq \rho \leq 1$.

A detailed study of the EO algorithm in its application to GBP and spin glasses is already provided in [26, 28], and we add only a number of minor modifications here. EO considers each vertex of a graph as an individual variable with its own fitness parameter. In the graph bi-partitioning, or for any other $0 < \rho \leq 1$, it assigns to each vertex i a 'fitness' $\lambda_i = -b_i$, where b_i is the number of 'bad' (unsatisfied) edges connecting i to other vertices. At all times an ordered list is maintained, in the form of a permutation Π of the vertex labels i, such that $\lambda_{\Pi(1)} \leq \lambda_{\Pi(2)} \leq \cdots \leq \lambda_{\Pi(N)}$, and $i = \Pi(k)$ is the label of the kth-ranked vertex in the list. In its most elementary version, EO forces sequential updates of the momentary worst variable $i = \Pi(1)$ at any update step, irrespective of the outcome, inducing a cascade of adaptive reorderings in the list. Since all variables occupy an identical and O(1)-sized state space, $\lambda_i = 0, -1, \ldots, -r$, for r-regular graphs, the list is highly degenerate and maintaining order or selecting variables (with fair tie-breaking rules) is done in O(1) computations.

To implement a local search of the configuration space, we must define a 'neighborhood' for each configuration within this space. At zero magnetization for all $0 \leq \rho \leq 1$, as an improvement over our previous implementation of EO for graph bi-partitioning, we proceed here by allowing imbalanced partitions up to a margin of ± 2 vertices, independently of the system size N. Then, we can pursue single-variable updates as long as the resulting configuration remains within the allowed imbalance. Valid ground states are only accepted if the current partition is perfectly balanced (although any O(1) imbalance for increasing N should result in identical scaling behavior). In this form, the single-flip neighborhood trivially generalizes to spin glasses with freely fluctuating magnetization, for which we simply ignore whether partitions remain balanced within the margins or not.

Much improved results are obtained with the following one-parameter implementation [25], called τ-EO: an integer $1 \leq k \leq N$ is drawn from a probability distribution $P(k) \propto k^{-\tau}$, $1 \leq k \leq N$, on each update, for fixed τ. Then, the vertex $i = \Pi(k)$ from the rank-ordered list of fitnesses is selected for an unconditional update. Over the course of a run, the cost of the configurations explored varies widely, since each update can result in better or worse fitnesses. In this form, EO maintains large fluctuations for overcoming barriers in the energy (or cost) landscape, while the selection *against* poorly adapted degrees of freedom ensures frequent returns to local minima. Thus, an EO run does not converge to a specific state. Instead, the output of a single run is the best configuration found along the way, which can be simply stored.

In this particular implementation of τ-EO, we use $t_{\max} = 0.1N^3$ update steps in each run. We choose at least three uncorrelated restarts for each instance here. The number of restarts is automatically adjusted for each instance. For example, should the current best results be first seen in run R, then the total number of runs is immediately increased to $\max(2R, 3)$. In this way, twice as many runs as were necessary to encounter the putative

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ground state for the first time are undertaken. Only at larger system sizes and degree r, when more than 10% of instances require more than those three restarts, do we initially set the duration of each run to up to $t_{\max} = 0.5N^3$ update steps, to again reduce the number of required restarts.

Note that no scales to limit fluctuations are introduced into the process, since the selection follows the scale-free power-law distribution over ranks $P(k)$, and since all moves are accepted. Instead of a global cost function, the rank-ordered list of fitnesses provides the information about optimal configurations. This information emerges in a self-organized manner, merely by selecting with a bias against badly adapted variables, rather than ever 'breeding' better ones. A theoretical analysis of the optimal τ value is discussed at length in [28, 25, 34, 29]; here, some initial trials suggest optimal values of $\tau = 1.2$–1.3, which we have used throughout.

Let us call $e_{\mathrm{GS}}(\rho, N)$ the ground state energy density, averaged over graphs and disorder in interactions, of (2) at magnetization fixed to zero, with ρ being the fraction of antiferromagnetic edges and N the graph size. Denote by $\tilde{e}_{\mathrm{GS}}(\rho, N)$ the same quantity at an arbitrary magnetization. We have obtained $e_{\mathrm{GS}}(\rho, N)$ on random regular graphs of degrees r between 3 and 10, and graph sizes between $N = 32$ and 1024. Statistical errors of our averages have been kept small by generating a large number of instances for each N and r, typically $n_I \approx 10^6$ for $N \leq 200$, $n_I \approx 10^5$ for $N \geq 256$.

All our data are indeed consistent with the conjecture that on sparse random regular graphs the ground state energy of (2) at zero magnetization is independent of ρ in the thermodynamic limit, $N \to \infty$. However, we also observe that the finite size corrections are dependent on ρ. This can be understood intuitively in the very particular case of random 2-regular graphs. A random 2-regular graph is basically a set of cycles of length $\sim \log N$. Hence, whereas the bisection width is at most two edges, the number of edges minus the max-cut size is of order $N/\log N$ (a unit cost for every other cycle in the graph). For $r \geq 3$, the finite size corrections are not that large, but they are quite different for different values of ρ.

In table 1, we compare the asymptotic ground state energy densities for different values of the graph degree r. The second column in this table presents the ρ-independent one-step replica symmetry breaking results for the ground state energy of (2) at zero magnetization, computed with the formalism developed in [35]. Note that the exact value for the ground state would be provided by the full-step replica symmetry breaking scheme which would give slightly larger values. The data in the third column are taken from [31]. In this case the magnetization was not fixed to be exactly zero on every instance, but it is zero in density in the thermodynamic limit. The data in [31, 32] are consistent with a power-law scaling

$$e_{\mathrm{GS}}(N) = e_{\mathrm{GS}} + aN^{-\omega} \tag{7}$$

with the value of the exponent $\omega = 2/3$ for all r. The finite size scaling for the graph bisection is clearly not consistent with $\omega = 2/3$, as is illustrated in figure 2 separately for odd and even values of r.[6] As was noted in [31, 33] ground state energies are strongly affected by the existence or absence of 'free spins' on graphs with purely even or odd degrees, respectively.

[6] Note, however, that due to possibly strong higher order corrections to (7) the values of ω in table 1 may be skewed.

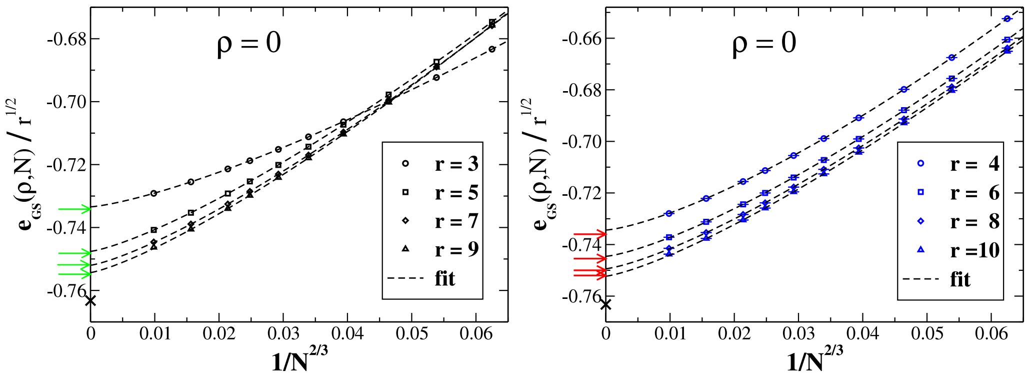

Figure 2. Plot of the rescaled average ground state energy densities, $e_{\mathrm{GS}}(N)/\sqrt{r}$, for the bi-partitioning of random regular graphs of degree r as a function of $1/N^{2/3}$. Errors are smaller than symbol sizes and are omitted for clarity. Nonlinear fits to each data set according to equation (7) are indicated by dashed lines. Arrows pointing to the ordinate (green for odd r, red for even r) mark the values extrapolated from table 1 (rescaled by $1/\sqrt{r}$) for the corresponding spin glass ground states from [31]. For large even and odd degree r, the extrapolated values approach the ground state energy of the Sherrington–Kirkpatrick model U_{SK} (black cross). The data are consistent with the conjecture that the spin glass asymptotic ground states are equal to the graph bi-partitioning ones.

On the left-hand side of figure 3, we show numerically the dependence of average ground state energies as a function of ρ at finite sizes on 3-regular graphs, only. While there are significant differences in magnitude of the corrections—even on average—for the smaller sizes, the finite size behavior soon becomes virtually independent of ρ and appears to converge towards the thermodynamic values found consistently at $\rho = 0$ and $1/2$ listed in table 1.

On the right-hand side of figure 3, we address the question on how finite size corrections are impacted by the constraint on magnetization being fixed to zero. We compare the average ground state energy of the antiferromagnet ($\rho = 1$) with magnetization strictly zero and with no constraint on the magnetization. Clearly, in the unconstrained case the distribution of ground state magnetizations is symmetrical above ρ_{c}, with vanishing fluctuations in the thermodynamic limit. Hence, the average magnetization must be zero. Yet, at finite size, it seems conceivable that fixing the magnetization makes otherwise lower energy states, i.e. ground states of the corresponding unconstrained model, unattainable, which may shift the ground state energies of the constrained system upwards. Indeed, as figure 3 demonstrates, while indistinguishable thermodynamically, average energies increase by large amounts relative to the unconstrained case ones, especially for small sizes. But the actual finite size corrections seem to be compatible with $\omega = 2/3$ in both cases. In our simulations, we generate r-regular graphs in such a way that multiple edges between identical vertices are not forbidden in their random assignment. Such multi-linked vertices have a probability of $\sim 1/N$ and, although noticeable at small size, do not affect any asymptotic scaling; see figure 3. Forbidding such edges makes it difficult to generate valid graphs especially at larger r and small sizes.

Table 1. Asymptotic ground state energy per spin for different values of the graph degree r. The second column, $e_{1\mathrm{RSB}}$, presents ρ-independent one-step replica symmetry breaking results [35]. The third column, $e_{\mathrm{GS}}(1/2)$, contains the numerical values of the extrapolated ground state energy for the spin glass [31], $\rho = 1/2$. The fourth and fifth columns give ground state energies $e_{\mathrm{GS}}(0)$ and scaling coefficients $\omega(0)$ for the graph bisection problem obtained from extrapolations to infinite graph size according to (7), as shown in figure 2. With minor exceptions for the smallest degrees, $e_{\mathrm{GS}}(1/2)$ and $e_{\mathrm{GS}}(0)$ are the same within error bars. Moreover, as we lack a theoretical justification for equation (7), the effective error bars of the fitted values are larger than those denoted in the table.

r	$e_{1\mathrm{RSB}}$	$\tilde{e}_{\mathrm{GS}}(\rho=\frac{1}{2})$	$e_{\mathrm{GS}}(\rho=0)$	$\omega(\rho=0)$
3	−1.272 31	−1.2716(1)	−1.2704(2)	0.89
4	−1.472 95	−1.472(1)	−1.469(1)	0.92
5	−1.675 20	−1.673(1)	−1.6717(5)	0.86
6	−1.829 17	−1.826(1)	−1.824(1)	0.87
7	−1.995 66	−1.990(3)	−1.990(1)	0.85
8	−2.126 81	−2.121(1)	−2.120(2)	0.87
9	−2.270 93	−2.2645(5)	−2.263(3)	0.85
10	−2.387 69	−2.378(3)	−2.379(4)	0.86

6. Possible generalizations

As we have argued above, the relation between max-cut and bisection width does not generalize (at least not straightforwardly) to the case of non-zero magnetization (not equally sized groups) or to the case of non-regular graphs. However, it does generalize to finite temperature properties of the Hamiltonian (2) at zero magnetization.

The relation of max-cut being the number of edges minus the bisection width also generalizes to the case of hypergraphs. In statistical physics, the corresponding models are known as models with p-spin interactions, in computer science as the XOR-SAT problem (the Boolean constraint satisfaction problem consisting of sets of linear equations).

An attractive but not (fully) valid generalization to discuss is the case of Potts spins $s_i = 1, \ldots, k$. The max-cut problem is then replaced by the 'max-k-coloring' problem and the bisection by k-partitioning. According to a result given by Kanter and Sompolinsky [36], analogous to that of [16], in dense graphs the two problems are related. Let pN be the degree of the graph, and k the number of colors; then according to [36] the maximum number of non-monochromatic edges is

$$|\mathrm{MaxCol}| = \frac{p}{2}N^2\left(1-\frac{1}{k}\right) + N^{3/2}\frac{|U(k)|}{k}\sqrt{p(1-p)}, \tag{8}$$

whereas the minimal number of edges between groups in the best balanced k-partition is

$$|k\mathrm{-part}| = \frac{p}{2}N^2\left(1-\frac{1}{k}\right) - N^{3/2}\frac{|U(k)|}{k}\sqrt{p(1-p)}. \tag{9}$$

Here, $U(k)$ is in both of the expressions for the ground state energy of the fully connected Potts model with k colors [37], given numerically in [36], while in the limit of large k it

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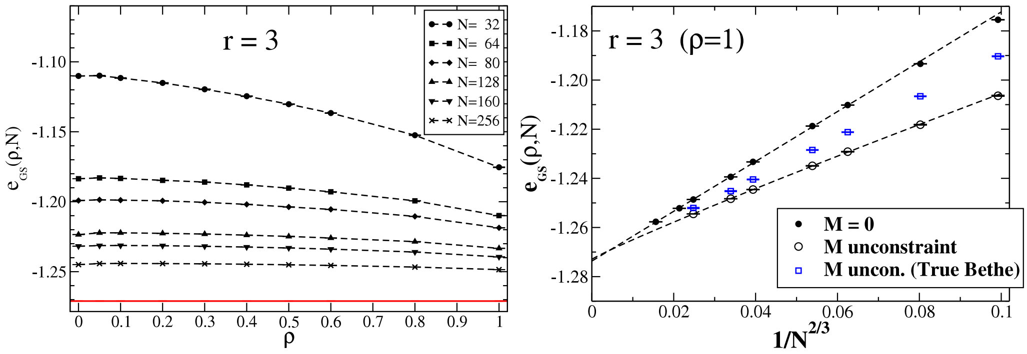

Figure 3. Left: the average ground state energy density for finite size 3-regular graphs as a function of the density of the antiferromagnetic bonds, ρ. The red dashed line marks the value extrapolated for $\rho = 0$ or $1/2$ from table 1. The data are consistent with the conjecture that the ground state energy is asymptotically independent of ρ. Right: plot of the average ground state energies for the purely antiferromagnetic spin model ($\rho = 1$) on 3-regular graphs as a function of system size, both at fixed ($M = 0$) and with unconstrained magnetization. The constrained case appears to extrapolate well linearly on a $N^{-2/3}$ scale without any transient, while the unconstrained case shows a small deviation of about 5% when all data are fitted according to equation (7). With blue squares, we also plot data for the same problem with unconstrained M but disallowing multiple edges between two vertices. All other data were obtained allowing graphs with such edges.

is $\lim_{k\to\infty} U(k) = \sqrt{(k \ln k)}$. For $k = 2$ the relations (8) and (9) reduce to the Fu and Anderson result (6).

On the basis of the analogy with the dense graph case, we would thus expect a generalization for sparse graphs, $p = c/N$, for $k > 2$ also. However, for three and more colors there is no apparent relation between the max-coloring and k-partitioning. Whereas there are some version of the Potts glass equivalent to the coloring problem (see e.g. [38]), there is no obvious gauge transform able to transfer the Potts ferromagnet (allowing 1 out of q values) to the Potts antiferromagnet (allowing $q - 1$ out of q values). This can be seen explicitly in the difference between the replica symmetric equations for the two problems. In the warning propagation sense [39], the neutral warning in max-coloring is created if two colors have the same value of an incoming field and the third one has a larger value. In contrast, in partitioning, the third value needs to be smaller. Also, the expressions for the replica symmetric energy are different, even after numerical evaluation. The equivalence thus holds only asymptotically in the first two orders of the degree of the graph (as suggested by the result in [36]). This underlines the exceptional nature of our main conjecture (1) for $k = 2$.

7. Conclusion

In this paper we describe, explain, and support with numerical evidence a conjecture that on sparse random regular graphs the ground state energy value of the spin glass Hamiltonian (2) for magnetization fixed to zero does not depend on the fraction of antiferromagnetic bonds. Although hints towards this conjecture can be found in the existing literature, we state it as a clear mathematical conjecture understandable for those non-specialist in spin glass theory: in random regular graphs, the asymptotic size of the max-cut equals the number of edges minus the minimal bisection width. We also summarize necessary conditions and limitations of this conjecture, in particular, that it does not generalize (at least not in a way that we could see) to non-regular graphs and non-zero values of the magnetization. We also support the conjecture with extensive numerical evaluations of the ground states. Finally, we believe that this paper will be useful to the mathematical and computer science community and that it will lead to a proof of this conjecture in the near future.

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References

[1] Garey M R and Johnson D S, 1979 *Computers and Intractability: A Guide to the Theory of NP-Completeness* (San Francisco, CA: Freeman)

[2] Alpert C J and Kahng A B, *Recent directions in netlist partitioning—a survey*, 1995 *Integr. VLSI J.* **19** 1

[3] Hendrickson B A and Leland R, *A multilevel algorithm for partitioning graphs*, 1995 *Proc. Supercomputing '95*

[4] Poljak S and Tuza Z, *Maximum cuts and large bipartite subgraphs*, 1995 *Combinatorial Optimization* (*DIMACS series in Discrete Mathematics and Theoretical Computer Science*) (Providence, RI: American Mathematical Society) pp 181–244

[5] Bollobas B, *The isoperimetric number of random regular graphs*, 1988 *Eur. J. Combin.* **9** 241

[6] Kostochka A V and Melnikov L S, *On bounds of the bisection width of cubic graphs*, 1992 *4th Czechoslovakian Symp. on Combinatorics, Graphs and Complexity* ed J Nesetril and M Fiedler (Amsterdam: Elsevier) pp 151–4

[7] Monien B and Preis R, *Upper bounds on the bisection width of 3- and 4-regular graphs*, 2001 *MFCS '01: Proc. 26th Int. Symp. on Mathematical Foundations of Computer Science* (London: Springer) pp 524–36

[8] Díaz J, Do N, Serna M J and Wormald N C, *Bounds on the max and min bisection of random cubic and random 4-regular graphs*, 2003 *Theor. Comput. Sci.* **307** 531

[9] Bertoni A, Campadelli P and Posenato R, *An upper bound for the maximum cut mean value*, 2006 *Graph-Theoretic Concepts in Computer Science* (*Lecture Notes in Computer Science*) (Berlin: Springer) pp 78–84

[10] Diaz J, Serna M J and Wormald N C, *Bounds on the bisection width for random d-regular graphs*, 2007 *Theor. Comput. Sci.* **382** 120 (Latin American Theoretical Informatics)

[11] Mézard M, Parisi G and Virasoro M A, 1987 *Spin-Glass Theory and Beyond* (*Lecture Notes in Physics* vol 9) (Singapore: World Scientific)

[12] Mézard M and Parisi G, *The Bethe lattice spin glass revisited*, 2001 *Eur. Phys. J.* B **20** 217

[13] Parisi G, *The order parameter for spin-glasses: a function on the interval 0–1*, 1980 *J. Phys. A: Math. Gen.* **13** 1101

[14] Talagrand M, *The Parisi formula*, 2006 *Ann. Math.* **163** 221

[15] Dembo A and Montanari A, *Ising models on locally tree-like graphs*, 2008 arXiv:0804.4726v2 [math.PR]

[16] Fu Y and Anderson P W, *Application of statistical mechanics to NP-complete problems in combinatorial optimization*, 1986 *J. Phys. A: Math. Gen.* **19** 1605

[17] Sherrington D and Kirkpatrick S, *Solvable model of a spin-glass*, 1975 *Phys. Rev. Lett.* **35** 1792

[18] Mézard M and Parisi G, *Mean-field theory of randomly frustrated systems with finite connectivity*, 1987 *Europhys. Lett.* **3** 1067

[19] Sherrington D and Wong K Y M, *Graph bipartitioning and the Bethe spin glass*, 1987 *J. Phys. A: Math. Gen.* **20** L785

J. Stat. Mech. (2010) P02020

[20] Wong K Y M and Sherrington D, *Graph bipartitioning and spin glasses on a random network of fixed finite valence*, 1987 *J. Phys. A: Math. Gen.* **20** L793

[21] Wong K Y M, Sherrington D, Mottishaw P, Dewar R and DeDominicis C, *Graph partitioning and dilute spin glasses: the minimum cost solution*, 1988 *J. Phys. A: Math. Gen.* **21** L99

[22] Montanari A, *Bisection of random graphs*, 2009 private communication

[23] Castellani T, Krzakala F and Ricci-Tersenghi F, *Spin glass models with ferromagnetically biased couplings on the Bethe lattice: analytic solution and numerical simulations*, 2005 *Eur. Phys. J.* B **47** 99

[24] Šulc P and Zdeborová L, *Belief propagation for graph partitioning*, 2009 arXiv:0912.3563v1 [cond-mat.dis-nn]

[25] Boettcher S and Percus A G, *Optimization with extremal dynamics*, 2001 *Phys. Rev. Lett.* **86** 5211

[26] Boettcher S, *Extremal optimization and graph partitioning at the percolation threshold*, 1999 *J. Phys. A: Math. Gen.* **32** 5201

[27] Boettcher S and Percus A G, *Nature's way of optimizing*, 2000 *Artif. Intell.* **119** 275

[28] Boettcher S and Percus A G, *Extremal optimization for graph partitioning*, 2001 *Phys. Rev.* E **64** 026114

[29] Hartmann A and Rieger H (ed), 2004 *New Optimization Algorithms in Physics* (Berlin: Wiley–VCH)

[30] Percus A G, Istrate G, Gonçalves B, Sumi R Z and Boettcher S, *The peculiar phase structure of random graph bisection*, 2008 *J. Math. Phys.* **49** 125219

[31] Boettcher S, *Numerical results for ground states of spin glasses on Bethe lattices*, 2003 *Eur. Phys. J.* B **31** 29

[32] Bouchaud J-P, Krzakala F and Martin O C, *Energy exponents and corrections to scaling in Ising spin glasses*, 2003 *Phys. Rev.* B **68** 224404

[33] Boettcher S, *Numerical results for ground states of mean-field spin glasses at low connectivities*, 2003 *Phys. Rev.* B **67** 060403

[34] Boettcher S and Grigni M, *Jamming model for the extremal optimization heuristic*, 2002 *J. Phys. A: Math. Gen.* **35** 1109

[35] Mézard M and Parisi G, *The cavity method at zero temperature*, 2003 *J. Stat. Phys.* **111** 1

[36] Kanter I and Sompolinsky H, *Graph optimization problems and the Potts glass*, 1987 *J. Phys. A: Math. Gen.* **20** L673

[37] Gross D J, Kanter I and Sompolinsky H, *Mean-field theory of the Potts glass*, 1985 *Phys. Rev. Lett.* **55** 304

[38] Krzakala F and Zdeborová L, *Potts glass on random graphs*, 2008 *Europhys. Lett.* **81** 57005

[39] Braunstein A, Mézard M, Weigt M and Zecchina R, *Constraint satisfaction by survey propagation*, 2003 *Computational Complexity and Statistical Physics* ed A G Percus, G Istrate and C Moore (Oxford: Oxford University Press) p 107

J. Stat. Mech. (2010) P02020