Life and times of an avalanche

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Abstract

The avalanche dynamics in the Bak–Sneppen mechanism of self-organized criticality (SOC) is discussed. Rigorous arguments reveal the emergent history dependence that dominates the intermittent activity. In numerical studies the emergent SOC state shows glassy features, such as aging and an ultrametric organization of activity. © 1999 Elsevier Science B.V. All rights reserved.

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Self-organized criticality (SOC) [1] describes a general property of slowly driven dissipative systems with many degrees of freedom to evolve intermittently with bursts spanning all scales up to the system size. Many natural avalanche-like phenomena have been represented using this concept, including earthquakes [2–5], biological evolution [6–9], and landscape formation [10]. Recently, SOC has been observed in controlled laboratory experiments on rice piles [11]. Theoretical models of rice piles [12] are related to a variety of different physical systems by universality [13]. A crucial ingredient for a system to exhibit SOC is the existence of thresholds that allow it to record the stress exerted by the driving force over long periods of time. We have demonstrated the emerging long-term memory analytically for the multi-trait model [14,15], a variant of the Bak–Sneppen model [9].

In the Bak–Sneppen model [9] species are sites on a lattice, representing a one-dimensional food chain. The fitness of each species is represented by a number between 0 and 1. Each species interacts with its neighbors. At each time step the “weakest” species with the lowest fitness value is “mutated”. This process is effectuated by replacing the fitness value with a random number drawn from a flat distribution between 0 and 1. This event in turn forces changes in the fitnesses of interrelated species. Their fitness landscape has changed by no fault of their own! In the model we simply replace

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the fitness values of all nearest neighbors with new random numbers from the same distribution. No matter what the initial state, the system evolves inevitably into a robust “critical” state in which correlations in space and time between events are distributed without any characteristic scale except for the total number \( N \) of species in the system \([16]\). In this SOC state, all but a few of the fitness values are distributed evenly above a certain critical fitness value \( \lambda_c \), leaving a gap below, as shown in Fig. 1. The fitness values above this threshold \( \lambda_c \) represent species that are unlikely to become the weakest species in the system any time soon. They have reached stasis, i.e. they are in an apparent equilibrium state that may only be punctuated when a weak neighboring species undermines their adaptation to the environment [“Punctuated Equilibrium” \([8]\)].

The species with fitness values in the gap below \( \lambda_c \) are the most active and constitute the avalanche. They are the most likely to become a global minimum soon and mutate. Species in the avalanche undergo a rapid sequence of changes until they and their neighbors collectively reach high fitness values and regain stasis. A critical avalanche ends (and a new one begins) when all species reach a fitness value at or above \( \lambda_c \).

The sequence of time intervals and the spatial extend of these avalanche events is power-law distributed.

In the multi-trait model \([14]\) the fitness of each species is dependent upon many traits associated with the different tasks that it has to perform. As in the Bak–Sneppen model above, a species is represented by a single site on a lattice. But the fitness of the collection of traits for each species is represented by a set of \( M \) numbers in the
unit interval. A larger number represents a better ability to perform that particular task, while smaller numbers pose less of a barrier against mutation. As before, at each time step there is one species which has one trait that has the lowest fitness in the entire system. Therefore, we “mutate” that one particular trait. Each neighboring species has one randomly chosen trait replace, since we assume that a mutation in the traits of one species can lead to a reevaluation of any one of the $M$ traits of a neighboring species. Thus, on a nearest-neighbor site, any trait has a $1/M$ chance to be updated. When $M \to \infty$, only a finite number of traits of any species populate the avalanche. Then, the model is solvable because fitness values in the avalanche evolve statistically independent from one another, because they can be eliminated only by becoming the global minimum, not through nearest-neighbor interactions. Still, even for $M = 1$ the coevolutionary chain reaction inevitably evolves to a self-organized critical state as in the Bak–Sneppen model, $M = 1$, and the full spatio-temporal complexity emerges.

To describe these spatio-temporal correlations at $M = \infty$, we define in Ref. [14] $F(r,s)$ to be the probability for an avalanche in the SOC state to survive precisely $s$ steps and to have affected a particular site of distance $r$ from its origin. We have $F \to 0$ for $r, s \to \infty$. Due to the statistical independence of active fitness values, one can find an exact evolution equation for $F(r,s)$, reminiscent of those describing slow dynamics in spin glasses [17]:

$$F(r,s) \sim \frac{1}{2} F_{rr}(r,s) + \frac{1}{4} \int_0^s ds' [2V(s-s') - F(r,s-s')] F(r,s') ds'$$  \hspace{1cm} (1)

with $V(s) = F(r=0,s) - 2\delta(s)$. In Ref. [14] this equation was used to show that the system becomes “critical” with power laws for the avalanche duration, $F(r = 0,s) \sim s^{-3/2}/\sqrt{s} (s \gg 1)$, and for the spatial extent of avalanches, $-\partial_r \sum_s F(r,s) \sim 24 r^{-3} (r \gg 1)$. Thus, $\tau = \frac{1}{2}$ and $\tau_R = 3$. Many other critical exponents can be determined explicitly [14]. In a long-lived avalanche each active site is visited many times, leading to punctuated equilibrium behavior. The distribution of first returns of the activity to a given site, $P_{\text{FIRST}}(s) \sim s^{-d_{\text{FIRST}}}$ for large $s$ in this model is found to be $\tau_{\text{FIRST}} = 2 - d/4$ ($d \leq 4$).

The integral kernel $V(s)$ in Eq. (1), which is long-range with a power-law tail only at the critical point, contains all of the history dependence of the process. With $V(s) \sim s^{-3/2}$, the probability to have reached a site at distance $r$ at time $s$ gets scale-free contributions from avalanches that reached $r$ at earlier times $s' < s$, representing the memory of the avalanche of previous activity over all time scales. The ultrametric tree structure of avalanches shows that they can be divided into sub-avalanches. Aavalanches contributing to $F(r,s)$ consist of sub-avalanches, one of which reaches $r$ in time $s'$ while the other’s combined duration is $s - s'$. The sub-avalanche structure gives a hierarchy of time scales. This changes the relaxation dynamics to be non-Gaussian, $F(r,s) \sim \exp[-C(r^D/s)^{(D-1)}]$ with $D = 4$. Our numerical results confirm a similar behavior for the Bak–Sneppen model with $D \approx 2.42$.

The history dependence of the avalanche dynamics in both, the Bak–Sneppen and the multi-trait model, leads to aging behavior reminiscent of glassy systems [18]. In
Refs. [19,20] we show that two-time autocorrelation functions $P(t_w, t)$, describing the return of activity to a site at time $t_a = t + t_w$ which was active most recently at time $t_a = t_w$ for an avalanche that started at $t_a = 0$, decay as power laws with two distinct regimes according to [21,22]

$$P(t_w, t) \sim t^{-\alpha} f \left( \frac{t}{t_w} \right), \quad f(x \ll 1) = 1, \quad f(x \gg 1) \sim x^{-r}.$$

(2)

The early time regime is that of the familiar stationary dynamics. The late time regime has a new critical coefficient $r$ characterizing non-stationary relaxation behavior of the SOC systems. The “waiting time” $t_w$ separating the early and late time regimes is a measure of the age of the avalanche.

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References