Broad universality in self-organized critical phenomena

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Abstract

We discover a broad universality class in self-organized critical phenomena. This universality class includes a model for dispersive transport in rice piles, the depinning transition of an elastic interface that is dragged at one end through a random medium, and the purely deterministic Burridge–Knopoff “train” model for earthquakes.

Keywords: Self-organized criticality; Universality; Interface depinning; Earthquakes; Granular materials

Self-organized criticality (SOC) \cite{cite1} as a means to describe correlated behavior of driven many-body systems has been controversial. Part of this stems from difficulties observing SOC in controlled laboratory experiments. Also, the existence of broad universality classes in SOC phenomena has been questioned. Demonstrating universality is clearly the first step toward a systematic mathematical theory.

Recently, evidence for SOC has been found in experiments by a group in Oslo on the granular dynamics of rice piles \cite{cite2}. The distribution of avalanche sizes measured for different system sizes was observed to obey finite size scaling with power law behavior. The Oslo group has also investigated tracer dispersion in the SOC pile by coloring rice grains and found that the average transport velocity of rice vanishes as the system size diverges. They proposed a “sand pile” model, herein referred to as the Oslo model, to describe their experiments \cite{cite3}.

We establish that a broad universality class exists for SOC phenomena. The Oslo “sand pile” model is mapped exactly to a model for interface depinning where the interface is slowly pulled at one end through a medium with quenched random pinning forces. The height of the interface maps to the number of toppling events in the sand pile model. The annealed noise of the random thresholds for toppling sand grains maps to quenched pinning forces for the interface. Thus a problem of dispersive transport \cite{cite4} in a granular medium can be recast in terms of the somewhat better understood problem of interface depinning \cite{cite5}. This leads to a number of scaling relations expressing critical exponents in the Oslo sand pile model in terms of the avalanche dimension $D$. This quantity is equal to the avalanche dimension for a uniformly driven interface, which has been determined numerically to be $D = 2.23 \pm 0.03$ \cite{cite5}. We also predict that the avalanche size distribution exponent $\tau = 2 - 1/D \simeq 1.55$ in the Oslo model, and that the average velocity of transport vanishes as $\langle v \rangle \sim L^{2-D} \sim L^{-0.23}$ when time is measured in units of the number of sand grains added. These predictions agree with previous

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numerical simulation results [3], and with our simulation results.

Finally, we conjecture that the Burridge–Knopoff [6] train model studied by de Sousa Vieira [7], a purely deterministic mechanical model with no embedded randomness of a spring–block chain pulled at one end, is also in the same universality class. Here an effective randomness at long length and time scales arises from the chaotic behavior of the deterministic chain due to friction.

The Oslo sand pile model is defined as follows: In a one-dimensional system of size \( L \), an integer variable \( h(x) \) gives the height of the pile at position \( x \), and \( z(x) = h(x) - h(x + 1) \) is the local slope. The boundary condition is \( h(L + 1) = 0 \). Grains are dropped at \( x = 1 \) until the slope \( z(1) > z^c(1) \); then the site topples and one grain is transferred to the neighboring site on the right \( x = 2 \). At each subsequent time step, all sites \( x \) with \( z(x) > z^c(x) \) topple in parallel. In a topping event at site \( x \), \( h(x) \rightarrow h(x) - 1 \) and \( h(x + 1) \rightarrow h(x + 1) + 1 \). No grains are added to the pile until the avalanche resulting from adding a sand grain ends and the system reaches a stable state with \( z(x) \leq z^c(x) \) for all \( x \). The annealed randomness describes in a simple way the changes in the local slopes observed in the rice pile experiments [3].

It is useful to define a local force

\[
F(x, t) = h(x, t) - h(x + 1, t) - \eta(x, H),
1 \leq x \leq L,
\]

where \( \eta \) are the randomly distributed critical slopes, i.e. \( \eta(x, H) = z^c(x) \) which take integer values 1 or 2 with equal probability. The boundary condition is \( h(L + 1, t) = 0 \) for all times. At each time step \( t \rightarrow t + 1 \), all unstable sites where \( F(x, t) > 0 \) topple. The quantity \( H(x, t) \) in Eq. (1) is the total number of topping events at site \( x \) up to time \( t \). The threshold slope at a site is chosen randomly after each topping event at that site; hence \( \eta(x, H) \) is an uncorrelated quenched random variable in the space of \((x, H)\). When all sites have reached a stable state where \( F(x) \leq 0 \), a grain of sand is added at site 1, \( h(1) \rightarrow h(1) + 1 \), \( t \rightarrow t + 1 \), and a new avalanche starts.

The number of sand grains at \( x \) at time \( t \) is the local gradient in the number of topplings that have occurred up to that time; \( h(x, t) = H(x - 1, t) - H(x, t) \). As a result, Eq. (1) can be rewritten as a dynamical equation for an interface with height profile \( H(x, t) \),

\[
F(x, t) = \nabla^2 H(x, t) - \eta(x, H), \quad 1 \leq x \leq L.
\]

This is an example of depinning of an elastic interface which has been widely studied [5,8]. The difference here is that rather than being driven uniformly, the interface is driven by being slowly dragged at the boundary.

The size of an avalanche, \( s \), for the interface is the integrated area during the burst resulting from pulling the end once. In the Oslo model, \( s \) is the total number of toppling events which occur after adding one grain of sand at the origin. The distribution of avalanche sizes is observed numerically to obey a scaling form [3]

\[
P(s) \sim s^{-\tau} G(s/L^D).
\]

In [9], we argue that \( D \) for the boundary driven interface, and hence for the Oslo sand pile model, is the same as for the uniformly driven interface. This hypothesis is confirmed by numerical simulations of the Oslo model giving \( D = 2.25 \pm 0.10 \) [3] and extremal interface depinning giving \( D = 2.23 \pm 0.03 \) [5]. Then we can relate all the other exponents in the Oslo model to the avalanche dimension \( D \) for interface depinning. In particular, due to conservation laws, we find \( \tau = 2 - 1/D \) and \( \langle \nu \rangle \sim L^{2-D} \).

Burridge and Knopoff [6] introduced a mechanical model for the stick–slip dynamics of earthquake faults. It consists of blocks connected by harmonic springs sliding with friction. The first element of the block–spring chain is connected to a driver that moves at constant velocity. It is referred to as the train model [7]. de Sousa Vieira found that it exhibits SOC. The train model is completely deterministic and contains no quenched randomness nor randomness in the initial conditions. The equation of motion for the position of the \( j \)th block, \( U_j \), is

\[
\ddot{U}_j = U_{j+1} - 2U_j + U_{j-1} - \Phi \left( \frac{\dot{U}_j}{\nu_c} \right).
\]
The static friction force $\Phi(0) = 1$, which is weakened at finite velocity

$$\Phi \left( \frac{\dot{U}}{v_c} \right) = \frac{\text{sgn}(\dot{U})}{1 + \dot{U}/v_c}. \quad (5)$$

The equation of motion is valid if the sum of elastic forces is greater than the static friction force; otherwise $\dot{U}_j = 0$. The train model has precisely one positive Lyapunov exponent giving chaotic behavior [10]. The blocks in this system exhibit slip–stick dynamics with a power law distribution of event sizes and extents.

We conjecture that the train model is in the same universality class as the Oslo model and boundary driven interface depinning. Since the model is dissipative and exhibits SOC, it is reasonable that a dissipative term $\dot{U}$ would dominate the acceleration term in Eq. (4) at long length and time scales. After a slip event, the blocks come to rest in a new configuration with a random elastic force increment at each site required to induce a subsequent slip event or toppling. This is the result of the chaotic dynamics, and in a coarse grained picture can be described by quenched random thresholds for static friction $\Phi(0)$ in the space of position and events, which corresponds to $\eta(x, H)$. Such an equivalence of a deterministic model with no embedded randomness which is chaotic with a stochastic model also occurs between the deterministic Kuramoto–Shivashinsky equation and the Langevin equation proposed by Kardar et al. [11]. Indeed, numerical simulations of the train model give $\tau - 1 \simeq 0.6$ and $\tau_R = 1 + D(\tau - 1) = D \simeq 2.2$ [7], which agree with the critical exponents measured for the Oslo model and support our conjecture.

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References