QUANTUM ENTANGLEMENT (WOW!)

Testing local realism with Freedman's inequality

CAUTION: A 20 mW violet laser is used in this experiment. Never look directly into the beam, including any reflected beams! Be mindful of accidental reflection off watches, jewelry, or other shiny objects! Avoid prolonged exposure of the skin to the beam, or it will begin to burn!

A RIDDLE

Suppose you flip a coin and hide it in your fist. When you open your hand, you see that the coin is heads. You conclude (correctly) that the coin was heads even before you opened your hand; opening your hand didn't change the state of the coin; the observation merely gave you information about the coin. But when you measure the polarization of a photon, do you detect the polarization that the photon had *all along*, or does the measurement fundamentally alter the state of the photon?

We might say that "hidden variables" (quantitative details about the way you flipped the coin and caught the coin) precisely determined the final orientation of the coin. We don't know the values of the hidden variables (that's why they're "hidden"), but in principle they could've been unhidden through sufficiently precise observations. Are there hidden variables in quantum mechanics, or prior to measurement, might a quantum state be fundamentally unknowable?

INTRODUCTION

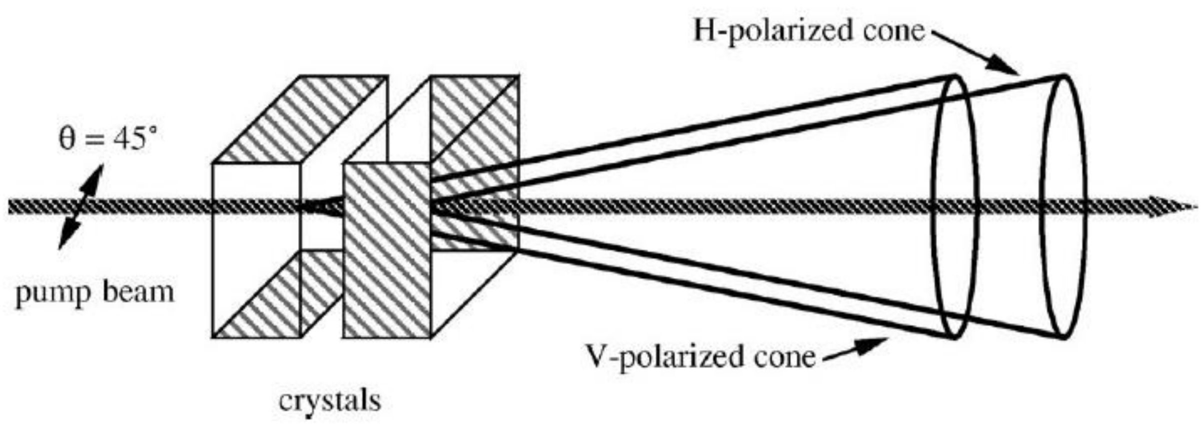

Figure 1. From Dehlinger and Mitchell, Am. J. Phys. **70**, 903-910 (2002). Pair of beta barium borate crystals. The thick arrow represents incoming 405 nm light. The cones represent 810 nm light produced when a 405 nm photon splits within a crystal.

We shine 405 nm violet light on a pair of beta barium borate (BBO) crystals, as shown in Figure 1. One of the BBO crystals is orientated to interact with horizontally polarized violet light, and the other is orientated to interact with vertically polarized violet light. If an incoming violet photon is **horizontally polarized**, it may split into two 810 nm infrared photons (at opposite sides of the cone) with **vertical polarization**. If an incoming violet photon is **vertically polarized**, it may split into two 810 nm infrared photons (at opposite sides of the cone) with **horizontal polarization**. This splitting of photons is called **spontaneous parametric downconversion**. We will set up our apparatus to detect infrared photons at two positions on opposite sides of the cone, as shown in Figure 2.

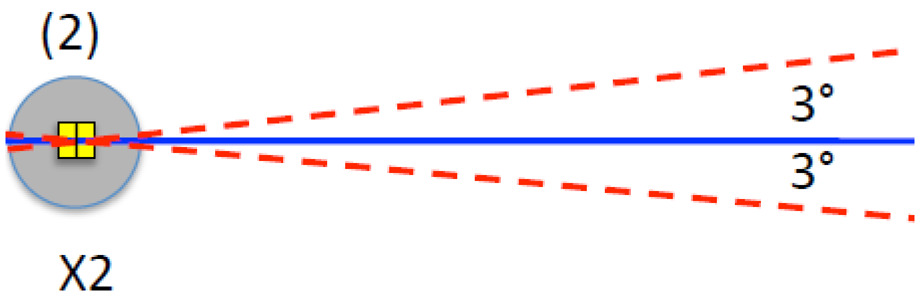

Figure 2. From Galvez (Colgate University). Horizontal "slice" of Figure 1. X2 is the crystal pair. The 810 nm photons travel at a 3° angle to the beam of 405 nm photons.

What happens when the incoming light is polarized at a 45° angle, as shown in the Figure 1? 45° polarization is a superposition of horizontal and vertical polarization. So each incoming photon has an equal probability of interacting with either of the crystals (but not both), and the resulting polarization of the of infrared photons is written

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle_A|H\rangle_B + |V\rangle_A|V\rangle_B\right), \tag{1}$$

where A represents the infrared photon at one side of the cone, and B represents the infrared photon at the other side of the cone. This expression means that the two photons were produced in the same crystal, but we don't know which one, so we don't know if the photons are horizontally or vertically polarized. But if measurement shows one photon to be horizontally polarized, the other one must be horizontally polarized as well. Likewise, if measurement shows one photon to be vertically polarized, the other one must be vertically polarized. The two photons are said to be entangled.

Imagine that we direct the two infrared beams toward polarizers. We want to determine the probability that the photons are transmitted, as a function of the angles of the polarizers. If a polarizer is at angle *a* relative to the vertical, we can imagine a polarization state |*a*> as a superposition of |V> and |H>:

$$|a\rangle = \sin a|H\rangle + \cos a|V\rangle. \tag{2}$$

You can derive this by simply decomposing |a> into vertical and horizontal components. If you're not familiar with |this notation>, just think of it as a way of representing a vector that may have horizontal and vertical components, equivalent to $\hat{\imath}\sin a + \hat{\jmath}\cos a$.

If the polarizer in front of A is at angle *a*, and the polarizer in front of B is at angle *b*, the **probability P(a,b) that <u>both photons</u> pass through the polarizers** is written

$$P(a,b) = \left| \langle a|_A \langle b|_B |\psi\rangle \right|^2 . \qquad (3)$$

In your lab report, you will prove that this equals **½cos²φ**, where **φ ≡ |*a*-*b*|**. Why does this result make sense? Suppose one photon reaches a polarizer slightly before the other. If the first photon goes through a polarizer at angle *a*, the other photon, for all practical purposes, immediately becomes polarized in the same direction. The second photon's probability of going through a polarizer at angle b is now given by Malus's law, $\cos^2\phi$. (Above, the factor of ½ occurs because the first photon is only 50% likely to pass through the first polarizer.)

We will discuss quantum entanglement and hidden variables below. First, let's just go through the details of the experiment.

EXPERIMENT

Every step must be completed meticulously! The laser beam is very narrow, and if the equipment is misaligned by even 1 mm, the experiment might not work!

1. Aligning the 405 nm laser

First you need to make sure that the 405 nm laser is at the same height as the BBO crystal pair, as shown in Figure 3.

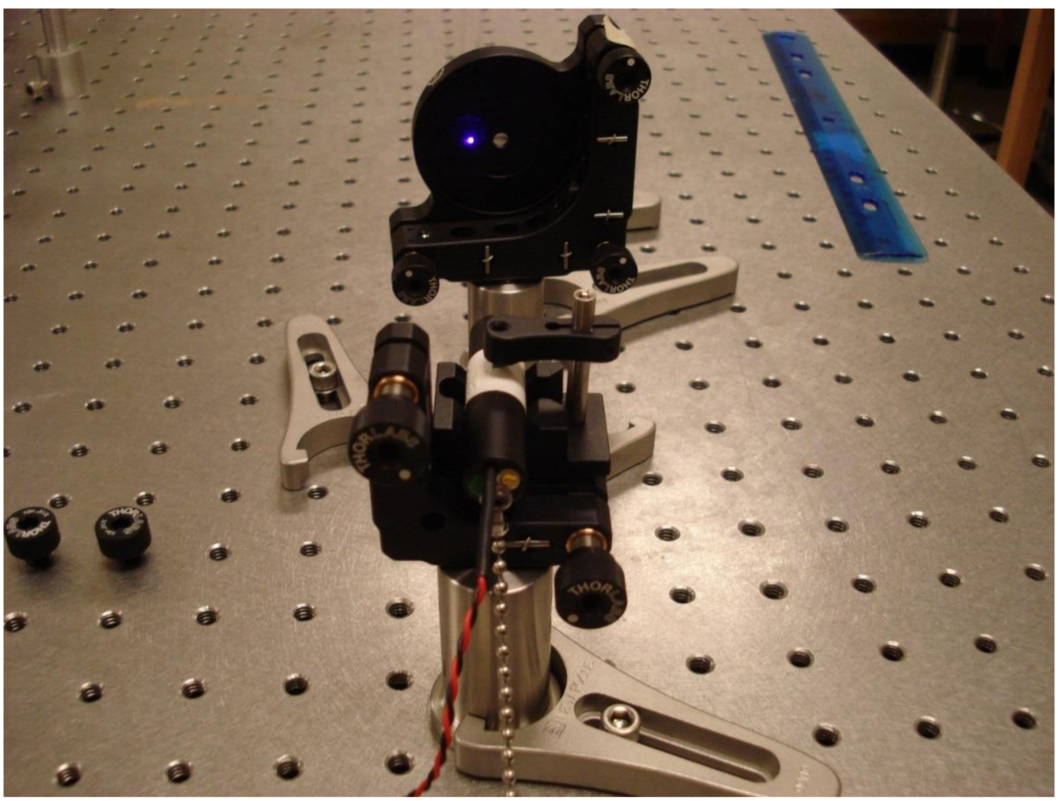

Figure 3. Making sure the 405 nm laser is at the same height as the crystal pair (the small clear circle in the center of the circular black mount).

You'll need to make the beam parallel to the table top, exactly above the row of holes leading to the center of the curved steel plate. To accomplish this, set two irises at exactly the height of the laser. Place an iris very close to the laser (Figure 4), and adjust the height of the iris until the beam goes straight through. Next, tighten a collar against the top of the post holder (Figure 5) holding the iris. This will preserve the height of the iris while allowing you to rotate the post within the post holder. (You will screw the post holders directly into the table, so the post holders won't be able to rotate. To rotate the post within the post holder, loosen the bolt in the post holder. The collar will preserve the height of the post.)

Repeat with the other iris.

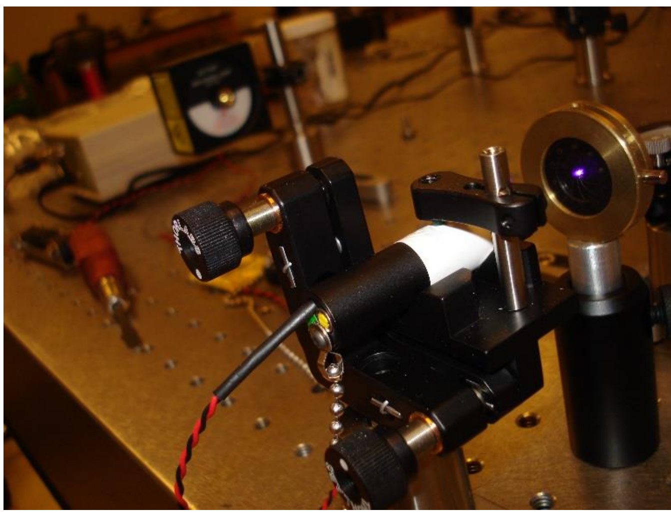

Figure 4. Setting iris to height of laser.

Figure 5. Placing collar to preserve height of iris.

In subsequent photos, the iris posts are held in shorter, less reliable post holders. Please continue to use the post holders shown in Figures 4 and 5.

Screw the post holder for one iris into the bolt hole where the crystal pair will go, 1 m from the steel plate (Figure 6).

Screw the post holder for the other iris into a hole along the same row of holes, near the steel plate (Figure 6).

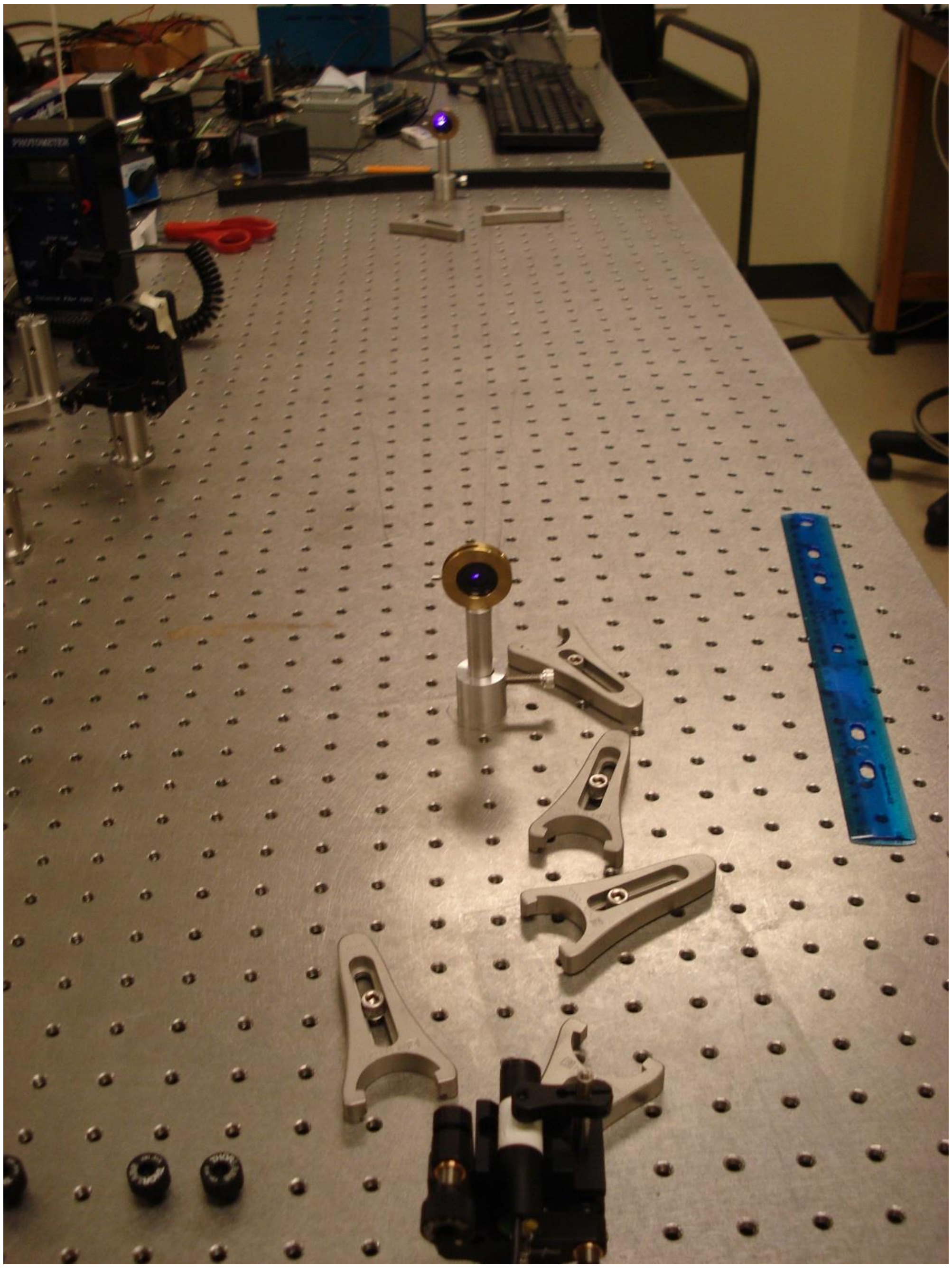

Figure 6. The irises are positioned over the row of holes leading to the center of the steel plate, and the 405 nm laser is aligned and tilted until its beam passes through both irises.

Adjust the position and tilt of the 405 nm laser until the beam goes through both irises. (Our laser produces a bright spot with a “bar” of light extending out in one direction. Ignore the bar; only align the bright spot.) Position the laser as close to the short edge of the table as you can, so it won’t get in the way of the mirrors you’ll soon set up. Clamp the laser in place.

2. Alignment of HeNe laser (along paths to be followed by invisible infrared photons)

Since the infrared beams are invisible, we will direct a red HeNe laser beam along the exact paths that the infrared beams will follow, to align the equipment the will detect the infrared photons.

Make sure the HeNe laser is at the same height as the 405 nm laser and irises (Figure 7).

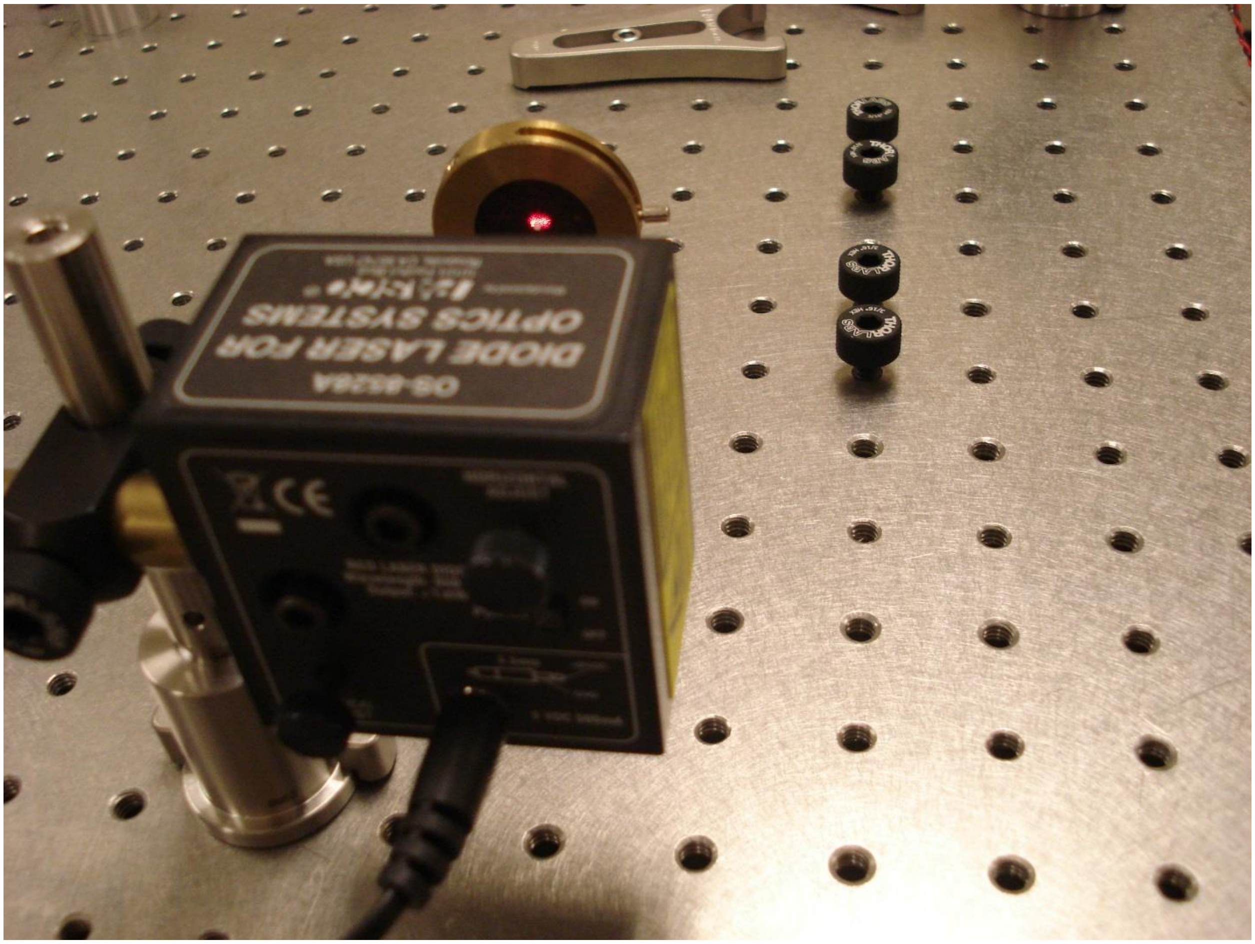

Figure 7. Place an iris near the HeNe laser and adjust the laser's height if necessary.

Adjust the vertical tilt of the HeNe laser if necessary so that its beam is perpendicular to the table; it must go through two irises at the correct height (Figure 8).

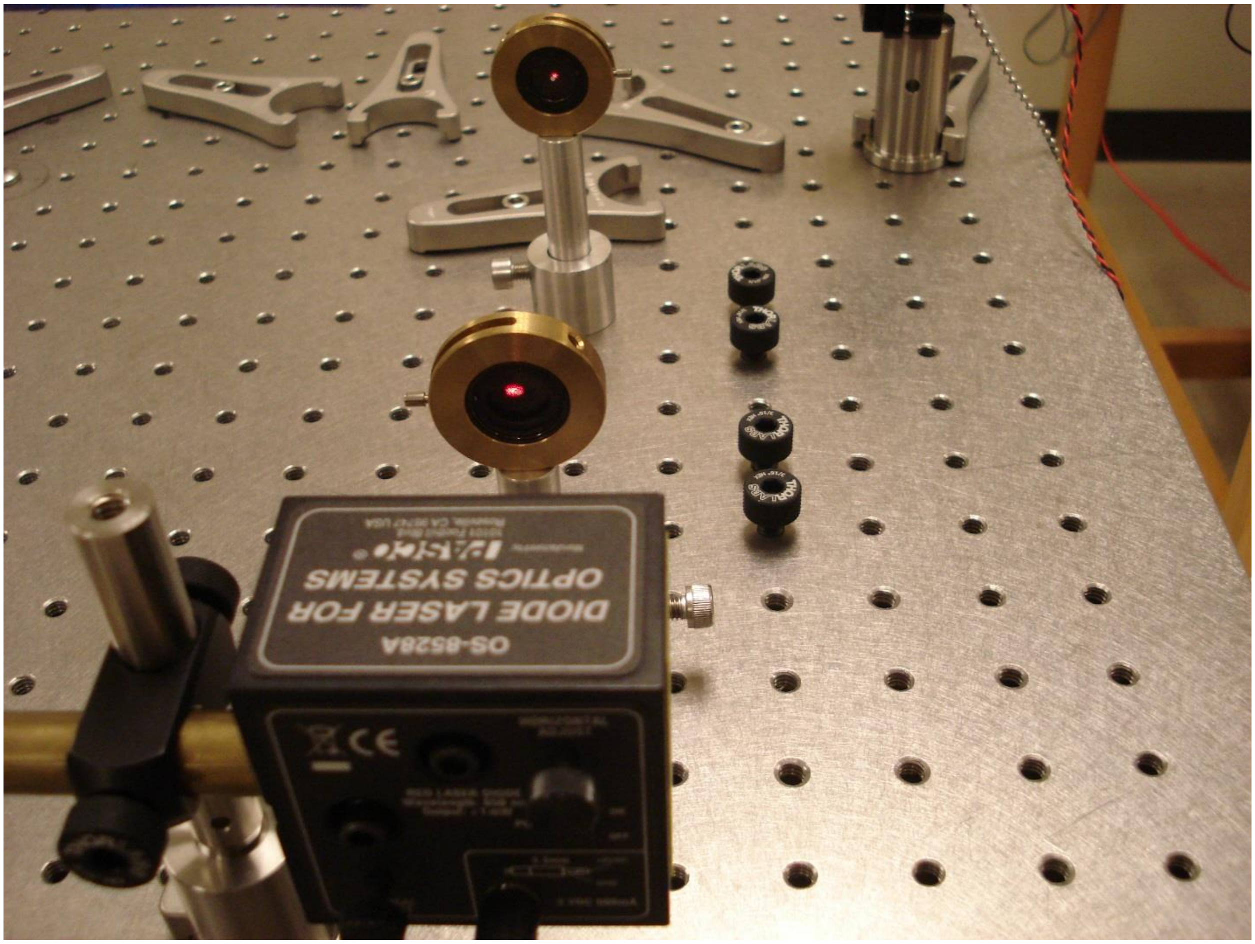

Figure 8. The HeNe beam must be at the same height as the 405 nm laser (and the two irises used with the 405 nm laser) and parallel with the tabletop.

Now position the HeNe laser and direct its beam roughly perpendicular to the 405 nm beam and approximately 3 inches in front of the 405 nm laser (Figure 9).

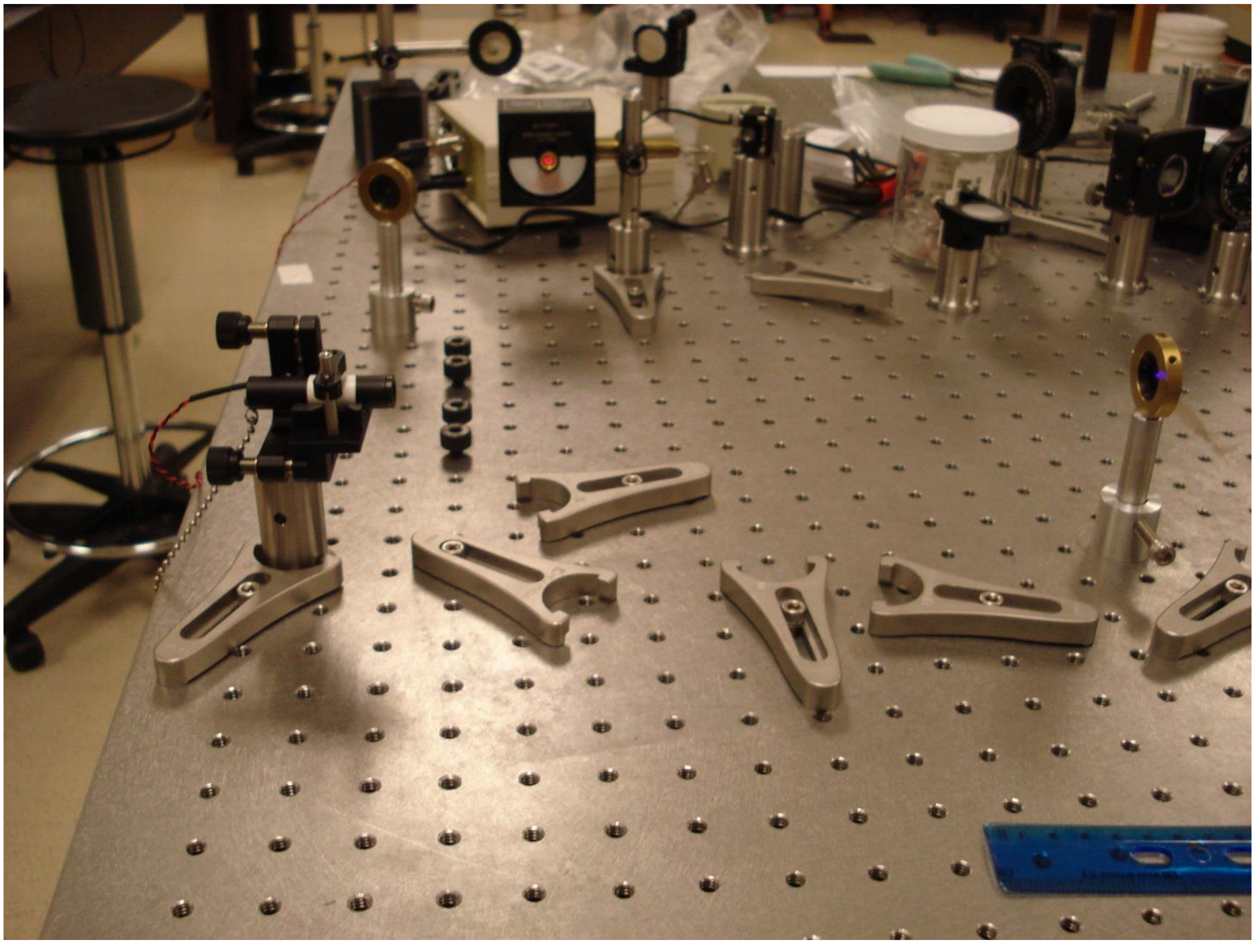

Figure 9. Setting up the HeNe beam perpendicular to the 405 nm beam.

Again have one iris where the crystal pair will go. Screw the post holder for the other iris into the bolt hole **38** inches towards the steel plate and **2** inches further from you (standing at the long edge), as shown in Figure 10. Arctan(2/38) = 3.01°, which is close enough to the 3°, the angle between the 405 nm beam and the infrared photons that will come out of the crystal pair.

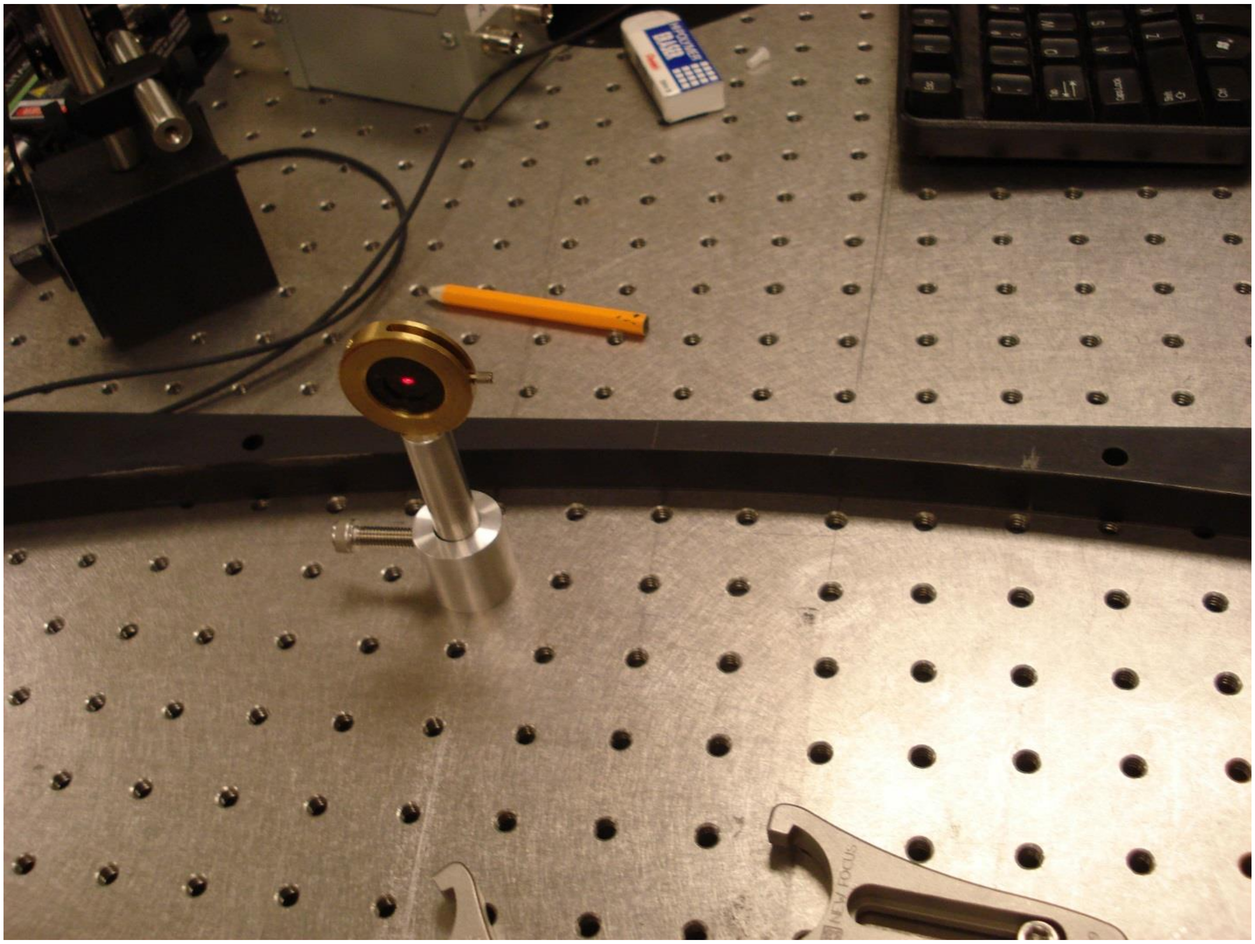

Figure 10. Positioning an iris along a path 3° from the 405 nm beam.

Position and tilt the mounted mirror so that the reflected HeNe beam goes through both irises (Figure 11). There is a risk that this mirror will get in the way of the next component. To avoid this risk, position the mirror so that the beam strikes it near its left edge, as viewed in Figure 11. Clamp the mirror in place.

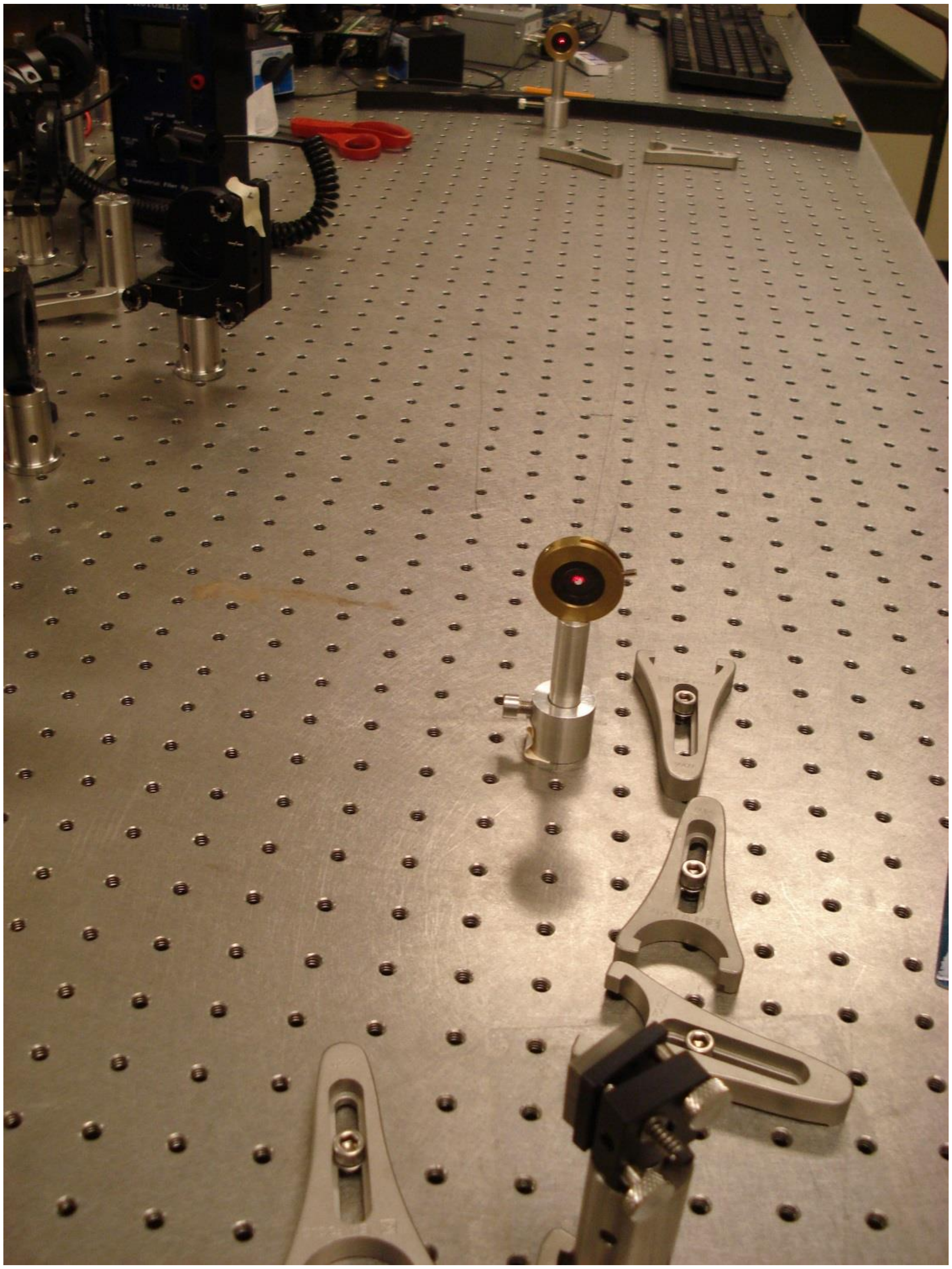

Figure 11. A mirror reflects the HeNe beam through both irises, 3° to the left of the 405 nm beam.

Now move the iris, which is by the steel plate, 4 inches so it's again 3° relative to the 405 nm beam, but in the opposite direction (Figure 12). Position and tilt the flipper mirror so the reflected HeNe laser goes through both irises. Clamp the flipper mirror in place. Remove the iris near the steel plate.

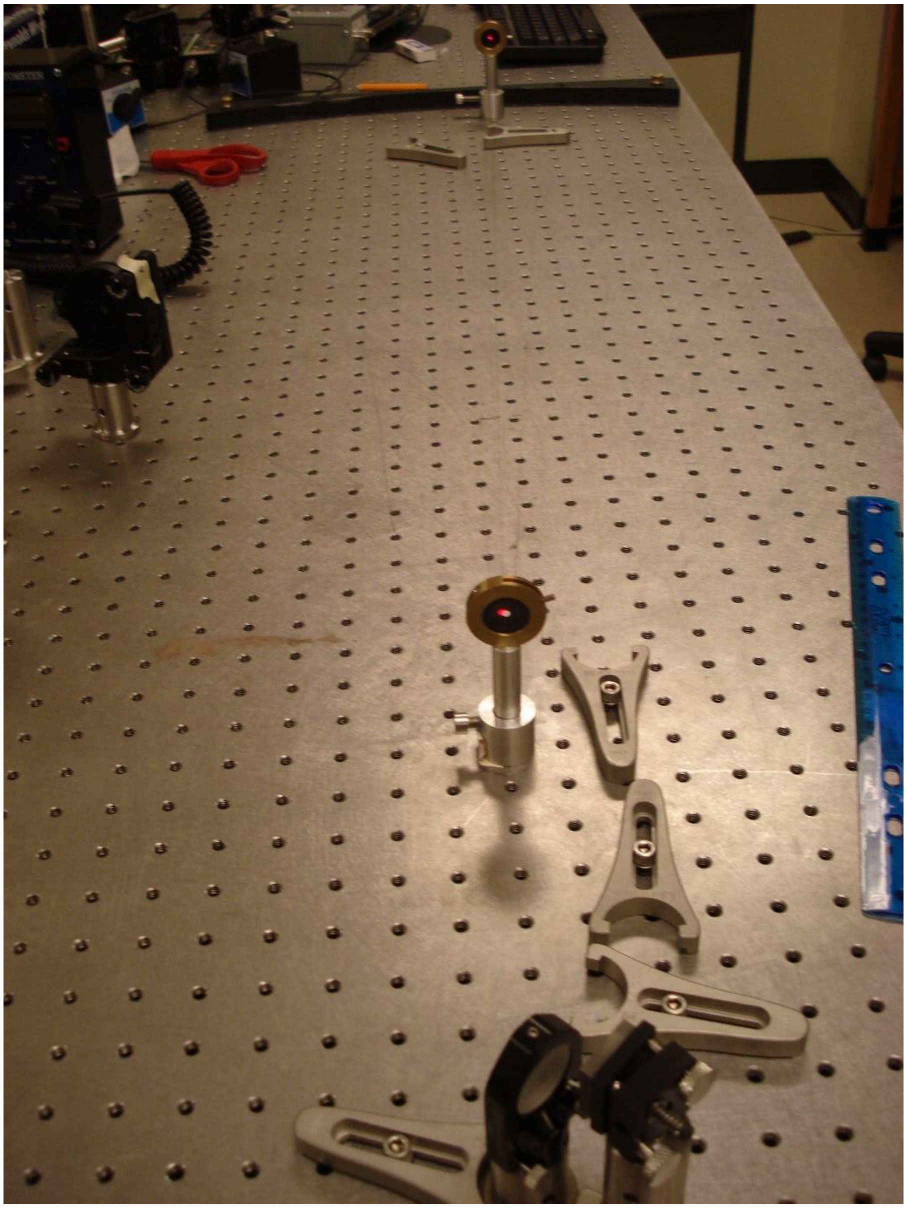

Figure 12. The flipper mirror must reflect the HeNe beam 3° to the right of the 405 nm beam.

3. Alignment of the collimators

The collimators are lenses that will focus the infrared beams onto fiber optic cables that lead to single photon detectors. Slide a collimator along the steel plate until the HeNe beam strikes the center of the iris on the collimator (Figure 13).

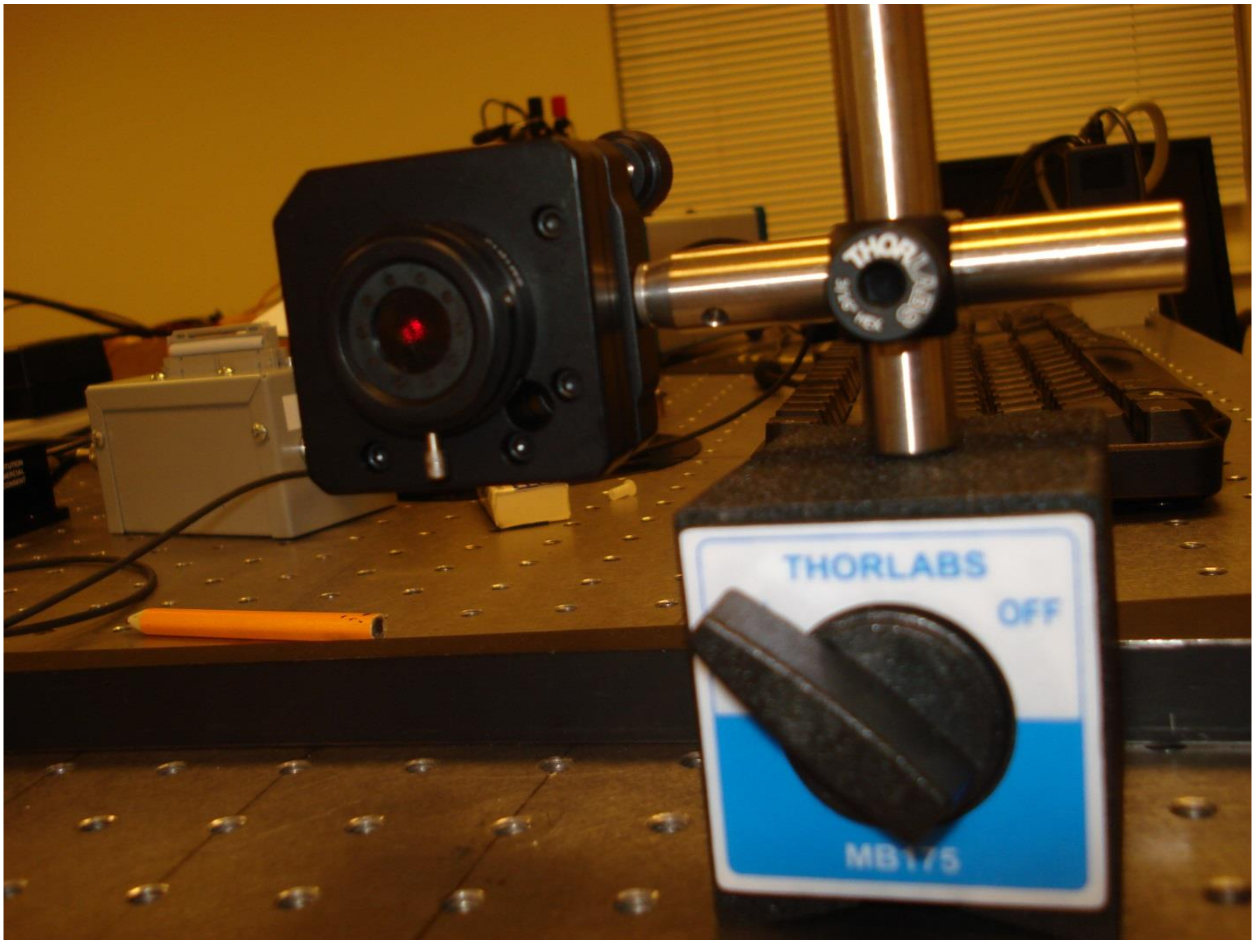

Figure 13. Aligning the collimator with the HeNe beam.

Now open that iris; we want all the light to enter the collimator.

Remove the fiber optic cable from the back of the collimator. Make sure the expanded red beam is continuing in the same direction as the HeNe beam into the collimator (Figure 14). If the expanded beam is misaligned laterally, slide the collimator along the track until the expanded beam is aligned correctly. The vertical position of the collimator may require adjustment as well.

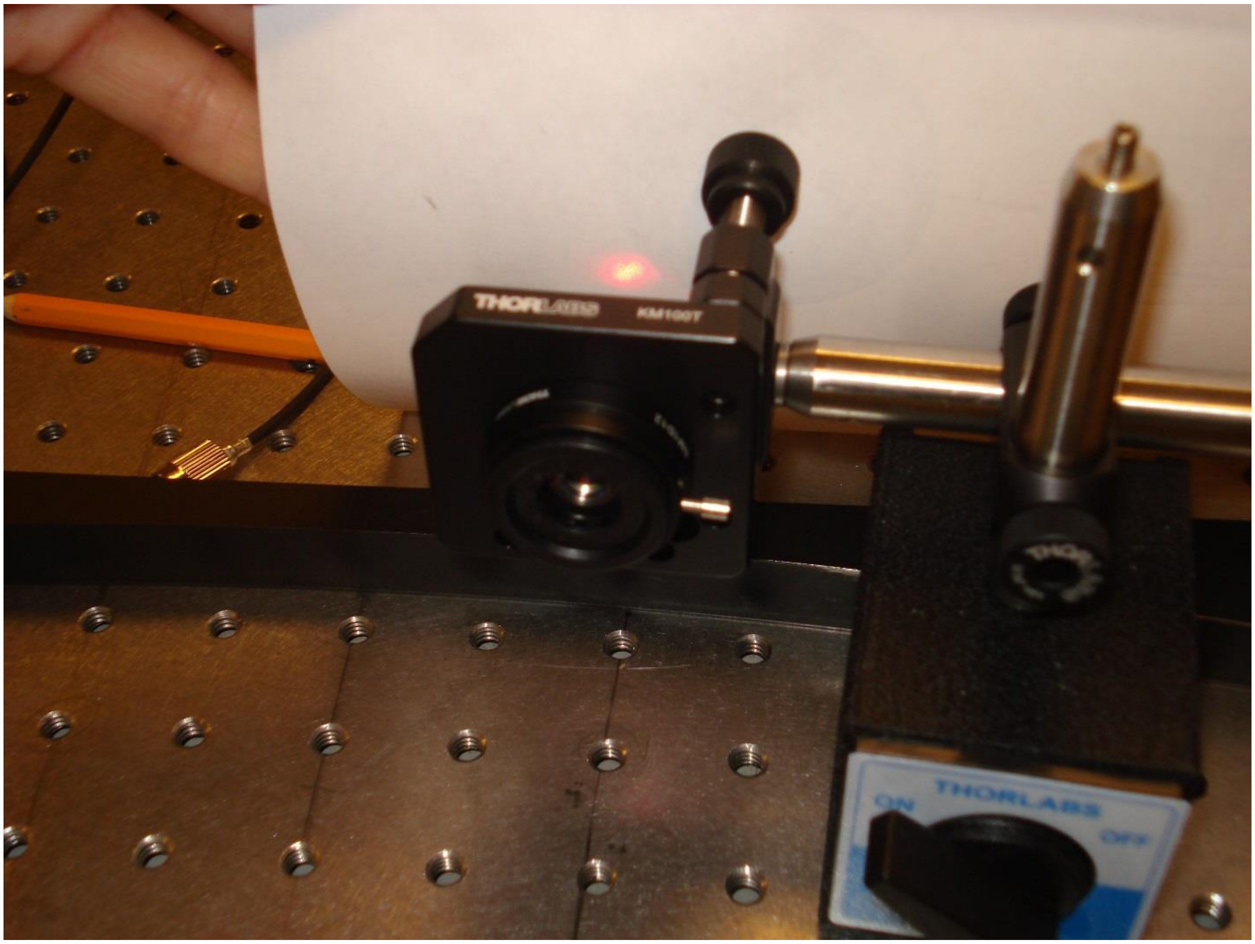

Figure 14. Observing the expanded beam behind the collimator.

Then hold the loose mirror against the front of the collimator (Figure 15).

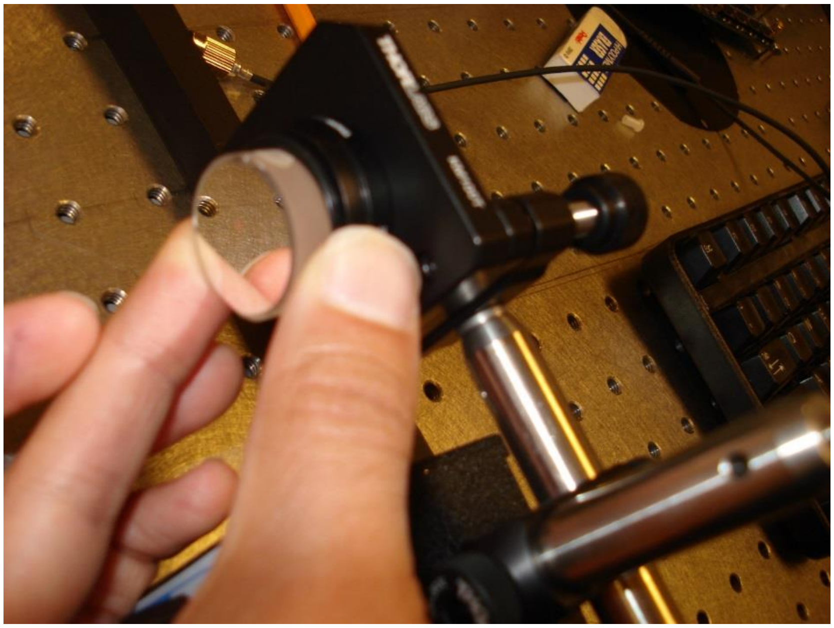

Figure 15. Placing the loose mirror against the collimator.

Adjust the tilt of the collimator until the reflected beam goes back through the iris where the crystal pair will go (Figure 16).

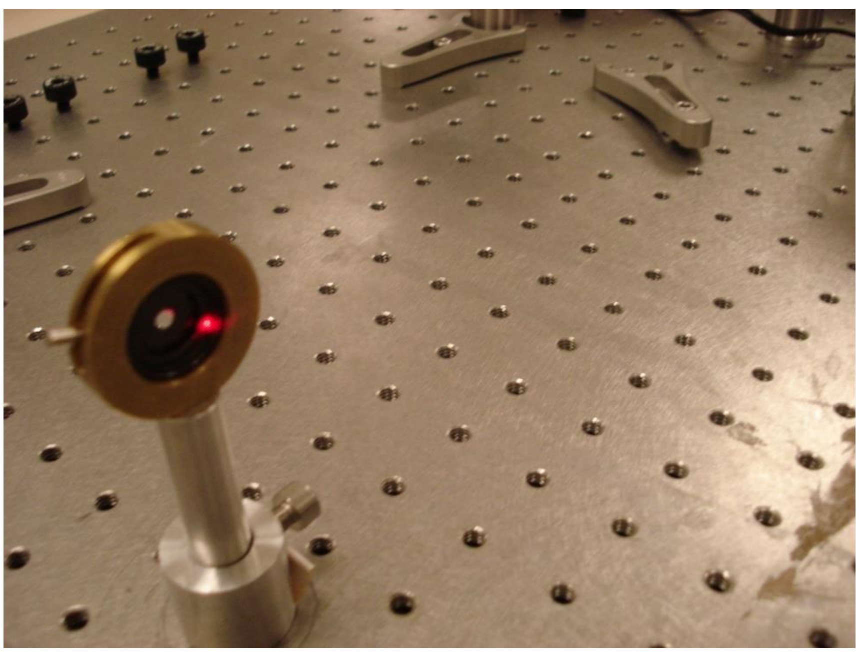

Figure 16. The tilt of the collimator must be adjusted until the beam reflecting off the loose mirror goes back through the iris.

Recheck the alignment of the expanded beam; adjust if necessary, and iterate these steps.

Reconnect the fiber optic cable to the collimator. Make sure it “clicks into place” before you tighten it; even before you tighten it, it shouldn’t be able to rotate. Disconnect the other end of the cable from the single photon detector. Observe the beam coming out of the fiber optic cable. When the cable is a few inches from the table, the beam should be bright and sharp (Figure 17). Readjust the tilt of the collimator to make the beam as bright and sharp as possible. Then reconnect the cable to the photon detector.

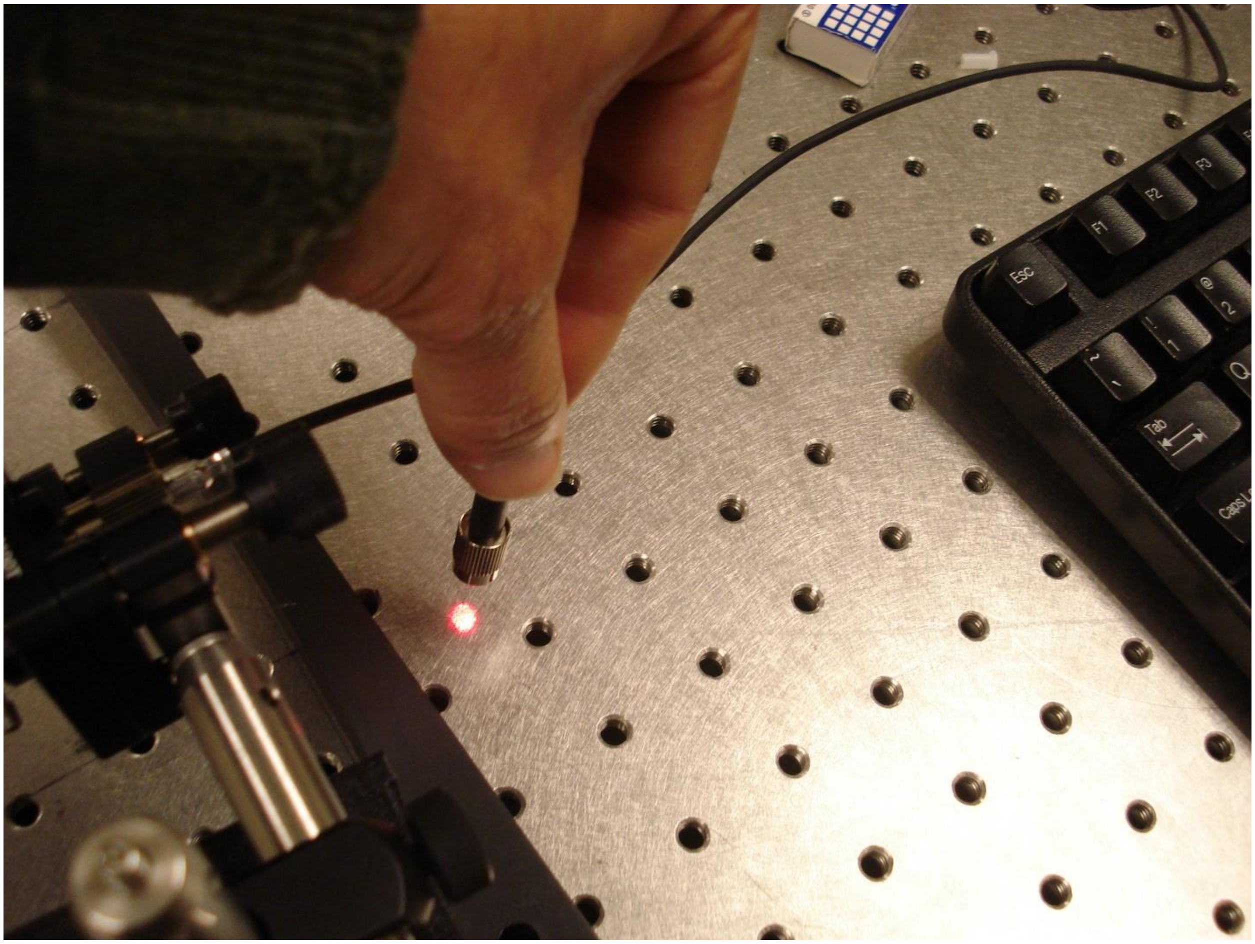

Figure 17. Observing the HeNe light coming out of the fiber optic cable.

Flip down the flipper mirror and repeat all the previous steps with the other collimator (Figure 18). When you're finished, unplug the HeNe laser. If all the alignment was done correctly, we don't need it anymore.

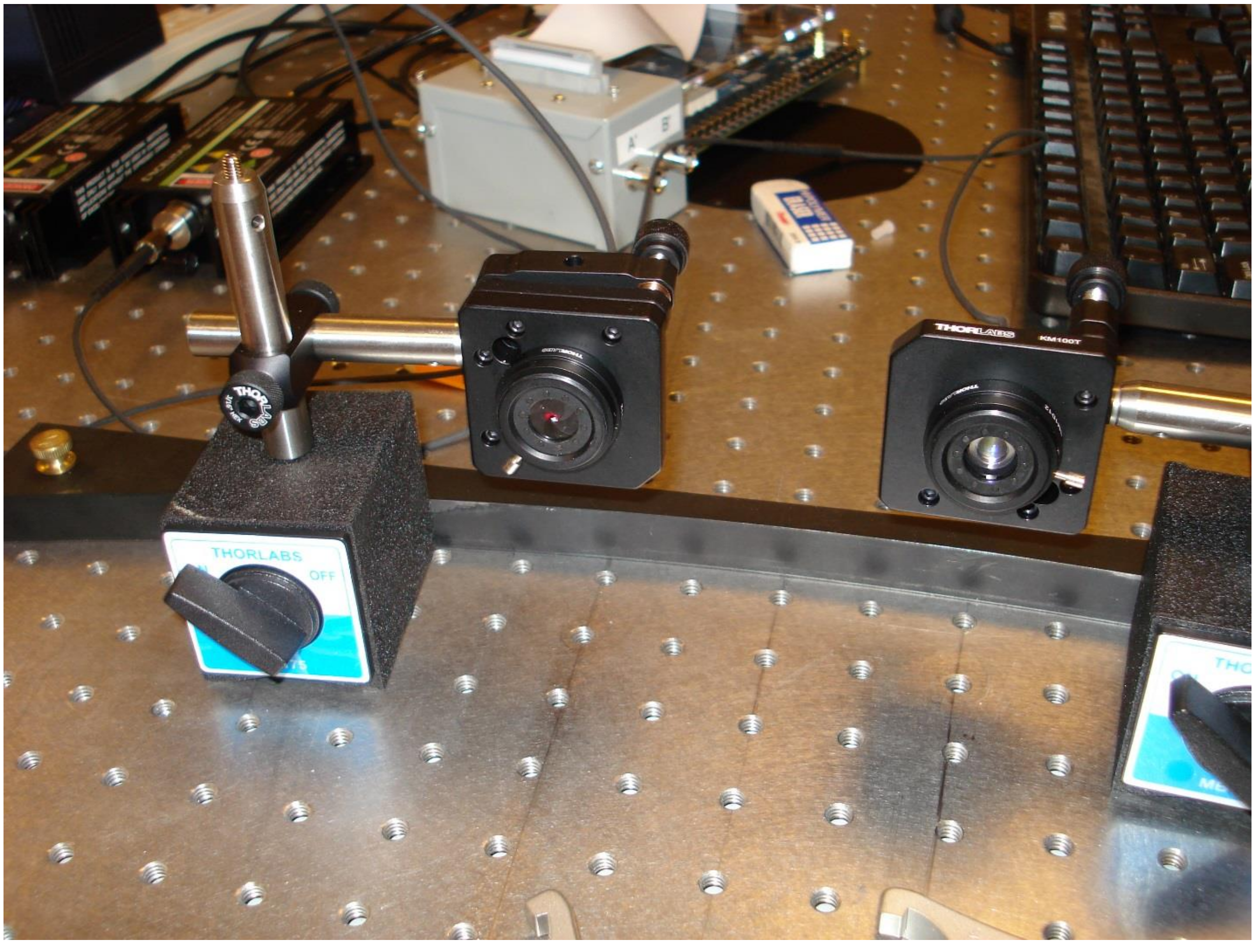

Figure 18. Repeat all the previous steps with the other collimator (flipper mirror flipped down).

4. Aligning the crystal pair

Remove the iris (and wide-headed bolt) and position the crystal pair exactly over the bolt hole (Figure 19).

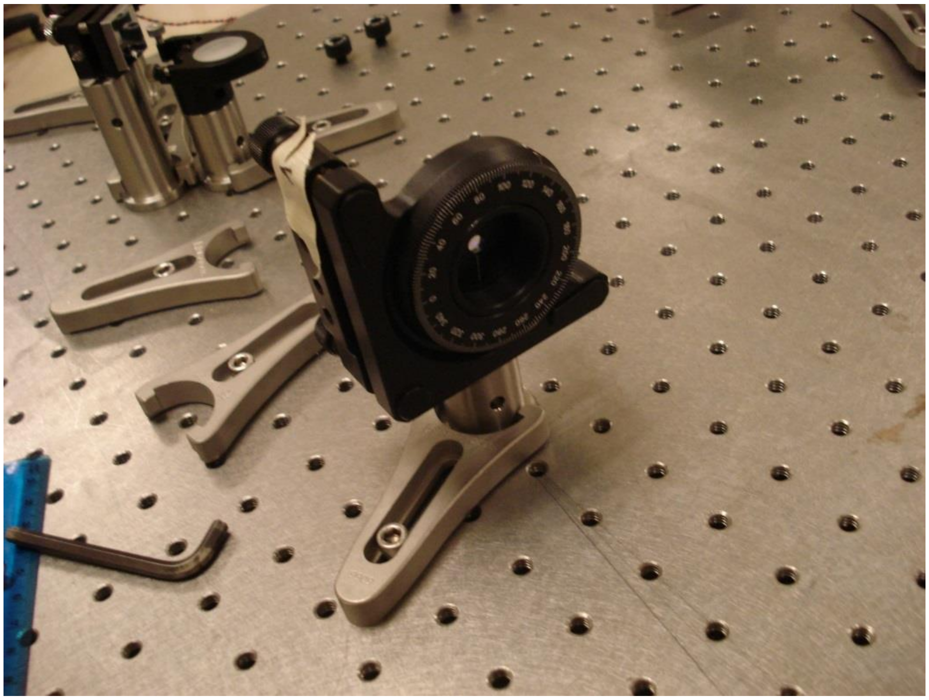

Figure 19. Clamp down the crystal pair so that the crystal pair (not the mount) is centered over the bolt hole.

We will adjust the tilt of the crystal pair separately for horizontally and vertically polarized light. The 405 nm laser is polarized, but it's difficult to rotate the laser while preserving its alignment. Therefore, we will send the 405 nm light through a half wave plate. (A half wave plate rotates the plane of polarization of linearly polarized light: the polarization is flipped about the "optical axis" of the half wave plate. As shown in Figure 20, if the optical axis makes an angle θ with the incoming polarization, the polarization is flipped 2θ.)

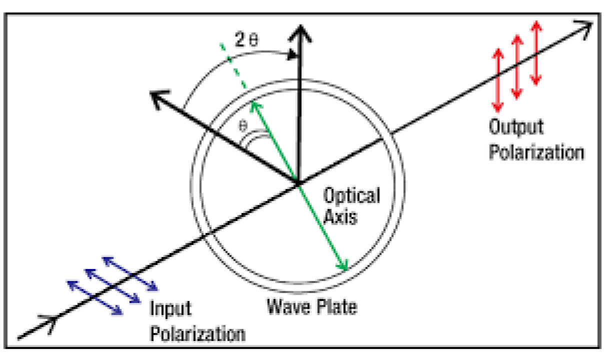

Figure 20. From Thorlabs. If the optical axis of the half wave plate makes an angle θ with the input polarization, the polarization is "flipped" by 2θ.

Place the 405 nm half wave plate between the laser and the crystal pair (Figure 21).

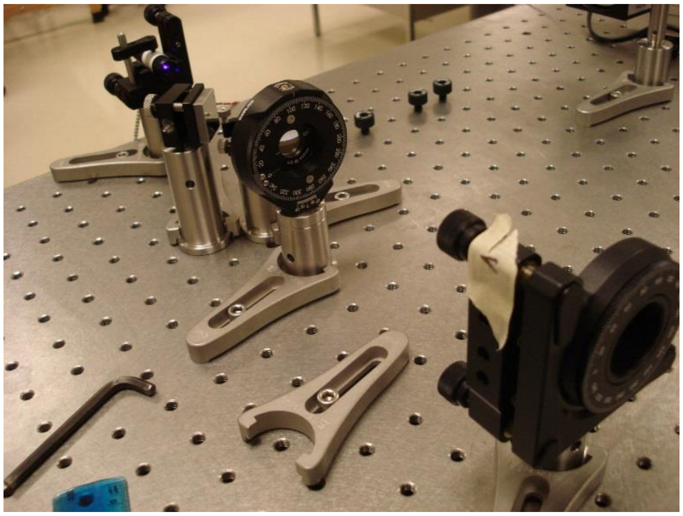

Figure 21. The half wave plate is placed between the laser and the crystal pair.

You can rotate the half wave plate to change the polarization of the 405 nm light. (You should send the light through a known polarizer and into a photometer to observe how final intensity depends on the angle of the wave plate.) Rotate the half wave plate so that the output polarization is horizontal. (When I last did this, I needed to set the half wave plate to 44°, but if someone rotated the laser in the holder, it's different now.)

Now place filters in front of the collimators to block as much visible light as possible (Figure 22).

Figure 22. These optical bandpass filters block all light except wavelengths near 810 nm. These do not have to be clamped in place.

Turn on the field programmable gate array (FPGA, Figure 23). The FPGA contains the electronic circuit that counts the voltages pulses produced by the single photons detectors. (The single photon detector outputs a voltage pulse every time a single photon arrives at the detector.)

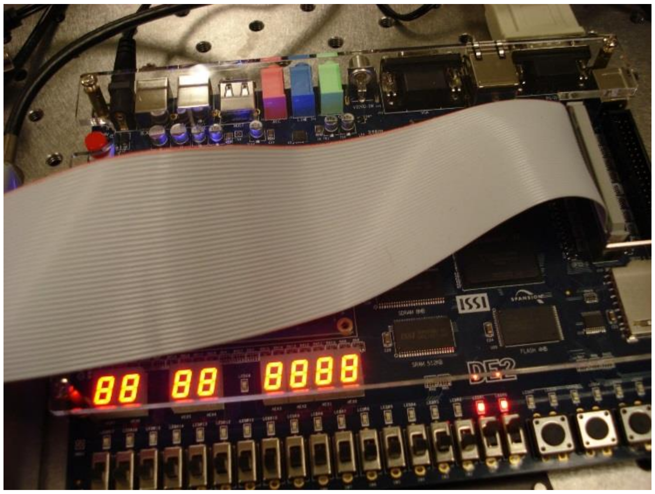

Figure 23. Push the red button to turn this device on. It contains that circuit that counts the voltage pulses output by the single photon detectors.

Open the Python or MATLAB script that will display the data.

TURN OFF THE ROOM LIGHTS BEFORE YOU TURN ON THE SINGLE PHOTON DETECTORS! THE SINGLE PHOTON DETECTORS WILL BE TEMPORARILY DISABLED AND POSSIBLY DESTROYED IF TURNED ON WHILE THE ROOM LIGHTS ARE ON! Once you're in the dark, turn on the power for the single photon detectors (Figure 24), but BE SURE TO TURN IT OFF BEFORE YOU TURN ON THE ROOM LIGHTS!

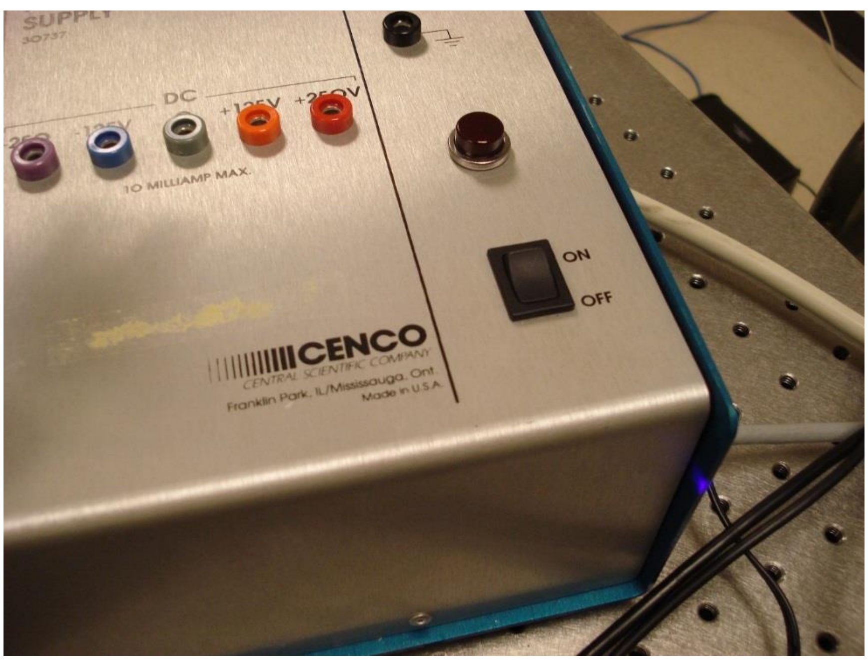

Figure 24. Turn this on **only in the dark! Always turn this off** before turning the room lights back on!

In the darkness, with the FPGA and single photon detectors turned on, run the MATLAB script. You should see three graphs: two showing counts/second for each of the two detectors (these are called “singles” counts), and also coincidences/second. A **coincidence** occurs when photons are detected **simultaneously** at the two detectors. **All we really care about is coincidences** because when a 405 nm photon splits in a BBO crystal, it produces two 810 photons which, if they’re in a horizontal plane, will arrive **simultaneously** at the two detectors.

If the alignment is good, you should be getting about 100 coincidences/second at this point. Adjust the **horizontal** tilt of the crystal pair to maximize coincidences. (We’re sending in horizontally polarized 405 nm light, which produces vertically polarized 810 nm light. It’s the horizontal tilt of the crystal pair that tilts the cone of vertically polarized 810 nm light.) As you adjust the tilt, the singles counts are affected as well as the coincidences. You should see both singles counts peak together.

If you’re getting about 100 coincidences/second (or more), you can leave well enough alone. If the coincidences are much lower, there are two other things you can try to increase coincidences: first, adjust the tilt of the collimators. Next, if you still want to try to increase coincidences, you can try slightly (by less than 1 mm) sliding one of the collimators along the steel plate.

We’ve adjusted the horizontal tilt of the crystal pair. Next, rotate the half wave plate to produce vertically polarized light (the half wave plate was at 359° when I did it). We want the coincidences to be the same as they just were for the other polarization. If they’re not, adjust the **vertical** tilt of the crystal pair until the coincidences are about the same for both polarizations.

5. Aligning the compensating crystal

Now we need the 405 nm photons to be polarized at 45° so that there are equal components of horizontal and vertical polarization. Rotate the 405 nm half wave plate to produce 45° polarization (I needed the half wave plate to be at 21.5°). Now place polarizers in front of the bandpass filters.

You should see that when both polarizers are vertical (0°), you get about the same coincidences as when they’re horizontal (90°). (If not, the crystal pair was not tilted properly.) Coincidences should be minimized when one polarizer is vertical and the other is horizontal. (Why? It’s important to understand this.)

Glance back (all the way back!) at Eq. (1). We need to make sure the horizontally polarized “component” is in phase with the vertically polarized component. However, horizontally polarized infrared photons are produced in one of the two crystals, and vertically polarized infrared photons are produced in the other one. One of the crystals comes before the other. The infrared photons produced in the first crystal must pass through the second crystal; those produced in the second crystal do not pass through any another crystal. This difference creates a phase shift that we preemptively reverse by use of a compensating crystal.

Place the compensating crystal between the 405 nm half wave plate and the crystal pair (Figure 25).

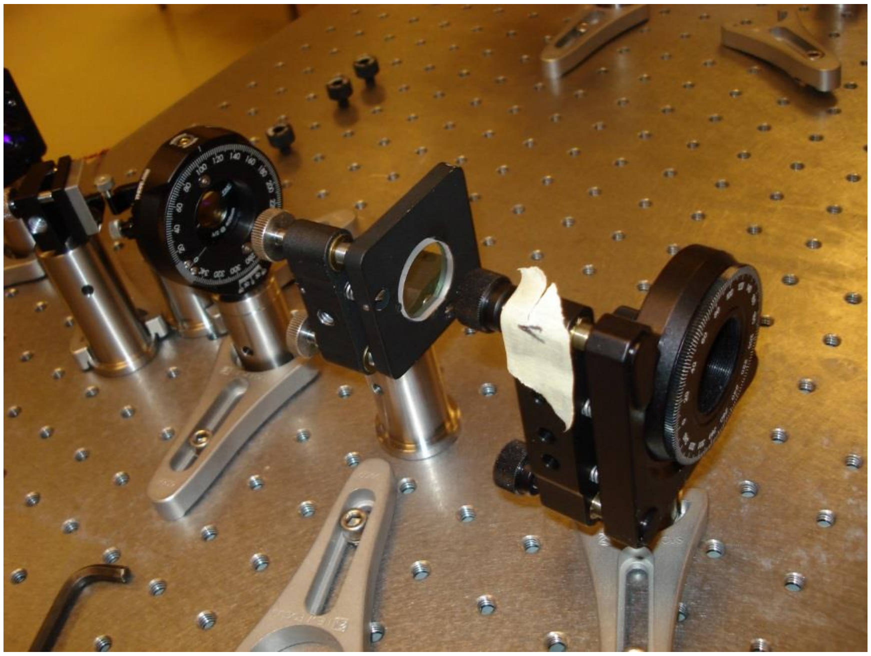

Figure 25. The compensating crystal goes between the half wave plate and the crystal pair. Remember, the room light must be off if the single photon detectors are on!

You've already achieved the following:

- Maximum and ~equal coincidences when both infrared polarizers are vertical or both horizontal.
- Minimum (~0) coincidences when one polarizer is vertical and the other is horizontal.

Adjust the tilt of the compensating crystal to achieve the following:

- Maximum and ~equal coincidences when the both polarizers are $a = b = 45°$ or $a = b = 135°$ from the vertical (same maximum as for both vertical, $a = b = 0$, or both horizontal, $a = b = 90°$).
- Minimum (~0) coincidences when one polarizer is 45° from the vertical and the other is 135° from the vertical.

I found that if I tilted the compensating crystal to get the same maximum at $a = b = 45°$ as I had for 0° and 90°, the other criteria were automatically satisfied. If all four bullet points are satisfied, you've achieved Eq. (1) and are ready to investigate quantum entanglement!

6. Test of a Bell inequality

We'll discuss the theory below. It's a little abstract, so I'll first describe the procedure, which is rather concrete and simple, now that you've completed the alignment.

Choose a time interval over which to record data. The larger the time interval, the smaller your uncertainties will be. 10 s is probably adequate.

Record coincidences for $\phi = |a-b| = 0°$, 11.25°, 22.5°, 33.75°, 45°, 56.25°, 67.5°, 78.75°, and 90°. For example, you can fix $\alpha = 0°$ and set β to the angles in the list. (You only need 22.5° and 67.5° for the Bell inequality, but it's nice to have the rest of the data so you can plot coincidences vs. ϕ.) Also record coincidences when both polarizers are removed. That's the whole experiment!

THEORY

In your lab report, you will derive an algebraic lemma: if real numbers x_1, x_2, y_1, y_2, X, and Y satisfy

$$0 \le x_1 \le X \tag{4a}$$

$$0 \le x_2 \le X \tag{4b}$$

$$0 \le y_1 \le Y \tag{4c}$$

$$0 \le y_2 \le Y, \tag{4d}$$

and

$$U \equiv x_1 y_1 - x_1 y_2 + x_2 y_1 + x_2 y_2 - Y x_2 - X y_1, \tag{5}$$

then it can be shown that

$$-XY \le U \le 0. \tag{6}$$

Now we proceed to the physics. Let N_{tot} be the total number of photon pairs arriving at the polarizers in a certain time interval. $N(a,b)$ represents the number of measured coincidences (simultaneous detection of two photons) within the same time interval, where a and b represent the angles of the two polarizers. For a sufficiently long time interval, the fraction of photon pairs detected (as coincidences) is the detection probability

$$P(a,b) = \frac{N(a,b)}{N_{tot}}. \tag{7}$$

We assume that each photon pair has a probability $p_{12}(\lambda,a,b)$ of detection, where λ is called a hidden variable, and the subscript is not "twelve" but represents detection of photon 1 and photon 2. λ and therefore the probability of detection may vary from one photon pair to another; Eq. (7) gives the average probability of detection over an ensemble of photon pairs. λ predetermines whether a particular photon pair is likely to be detected. We could consider the special case in which $p_{12}(\lambda,a,b)$ is always 0 or 1, such that each photon pair's fate (coincidence detection or nondetection) is predetermined with complete certainty: the photons all along have definite properties that the measurement merely unveils. This assumption is called *realism*. We're actually considering a much broader class of hidden variable theories in which $p_{12}(\lambda,a,b)$ may have intermediate values (between 0 and 1).

$\rho(\lambda)$ is the probability distribution of λ, so that

$$P(a,b) = \int \rho(\lambda) p_{12}(\lambda, a, b) d\lambda. \tag{8}$$

Eq. (8) is a bit abstract, so let's think about it. λ determines $p_{12}(\lambda,a,b)$, the probability of detecting a photon pair for a choice of polarizer angles. λ may vary from one photon pair to another because the detection probability may vary from one photon pair to another. When we average over all detection probabilities for individual pairs, we get the overall detection probability, P(a,b).

If we assume that the measurements of the two photons are independent events, then $p_{12}(\lambda,a,b)$ is the product of the probabilities of each separate measurement: $p_{12}(\lambda,a,b) = p_1(\lambda,a,b)p_2(\lambda,a,b)$. We further assume that a photon's probability of transmission through a polarizer depends on the angle of that polarizer, but not on the angle of the polarizer in front of the other photon. The combination of these assumptions is called *locality*: the measurement of one photon depends only on that photon and the polarizer it encounters.

We specify that a is the angle of the polarizer encountered by photon 1, and b is the angle of the polarizer encountered by photon 2. Assuming locality, $p_1(\lambda,a,b)$ simplifies to $p_1(\lambda,a)$, $p_2(\lambda,a,b)$ simplifies to $p_2(\lambda,b)$, and

$$p_{12}(\lambda,a,b) = p_1(\lambda,a)p_2(\lambda,b). \tag{9}$$

Let $p_1(\lambda,\infty)$ represent the probability of detecting photon 1 when the polarizer in front of it is removed. There is no known mechanism through which the presence of the polarizer can increase the number of detected photons. Therefore,

$$0 \le p_1(\lambda,a) \le p_1(\lambda,\infty). \tag{10a}$$

For any other polarizer angle a',

$$0 \le p_1(\lambda,a') \le p_1(\lambda,\infty). \tag{10b}$$

Similarly,

$$0 \le p_2(\lambda,b) \le p_2(\lambda,\infty) \tag{10c}$$

and

$$0 \le p_2(\lambda,b') \le p_2(\lambda,\infty). \tag{10d}$$

Applying the lemma of Eqs. (4)-(6),

$$\begin{aligned}
&-p_1(\lambda,\infty)p_2(\lambda,\infty) \le \\
&p_1(\lambda,a)p_2(\lambda,b) - p_1(\lambda,a)p_2(\lambda,b') + p_1(\lambda,a')p_2(\lambda,b) + p_1(\lambda,a')p_2(\lambda,b') - p_1(\lambda,a')p_2(\lambda,\infty) - p_1(\lambda,\infty)p_2(\lambda,b) \\
&\le 0
\end{aligned} \tag{11}$$

When Eq. (11) is multiplied by $\rho(\lambda)d\lambda$ and integrated, and each term is simplified through Eq. (8), then

$$-P(\infty,\infty) \le P(a,b) - P(a,b') + P(a',b) + P(a',b') - P(a',\infty) - P(\infty,b) \le 0. \tag{12}$$

We next assume rotational invariance in the photon pairs so that the measured coincidence count depends only on the angle $\phi = |a\text{-}b|$ between the polarizers: $P(a,b) = P(\phi)$. (Rotational invariance means that nothing changes when you rotate both polarizers through the same angle in the same direction.) You can test this assumption experimentally: keep ϕ constant while changing *a* and *b*. You'll show in your lab report that Eq. (1) is (perhaps surprisingly) a rotationally invariant state, according to quantum theory.

Then choosing *a*, *a'*, *b*, and *b'* to satisfy

$$|a-b| = |a'-b| = |a'-b'| = |a-b'|/3 = \phi, \tag{13}$$

Eq. (12) simplifies to

$$-P(\infty,\infty) \leq 3P(\phi) - P(3\phi) - P(a',\infty) - P(\infty,b) \leq 0. \tag{14}$$

In your lab report, you will show how to simplify Eq. (14) to

$$-N_0 \leq 4N(22.5°) - 4N(67.5°) \leq N_0, \tag{15}$$

where $N_0 \equiv N(\infty,\infty)$ is the number of measured coincidences when both polarizers are removed, and $N(\phi)$ is the number of coincidences when the angle between polarizers is ϕ. Eq. (15) simplifies to

$$\delta \equiv \left|\frac{N(22.5°) - N(67.5°)}{N_0}\right| - \frac{1}{4} \leq 0, \tag{16}$$

This version of a Bell inequality was first derived by Freedman. If valid assumptions are made in the derivation of the inequality, δ must be nonpositive. If measurement contradicts this requirement, then one or more of the assumptions must be incorrect. We assumed that the state of each photon was determined all along by a hidden variable and was not influenced by the measurement of the other photon—this assumption must be false.

Due to experimental uncertainty, a positive δ does not necessarily guarantee a violation of Eq. (16): the result is inconclusive if σ_δ, the uncertainty in δ, is greater than δ itself. So it is critical to understand error and error propagation. The idea is that if we repeat the experiment many times, we obtain a different value of δ each time. There will be a certain spread, or standard deviation, in the δ values. σ_δ is the standard deviation in δ, and we think of it as the uncertainty, or estimated error, in δ. (In other words, for our purposes, these three terms are interchangeable: error, uncertainty, and standard deviation.) σ_δ depends on σ_{N_0}, $\sigma_{N(22.5°)}$, and $\sigma_{N(67.5°)}$, the uncertainties (standard deviations) in the measured coincidence counts.

If you measure N_0 many times, you will not get the same result every time. Assuming the many values of N_0 follow a Poisson distribution, **the standard deviation in N_0 is the square root of the average value of N_0**. Instead of actually measuring N_0 many times, we may choose to measure it just once. We then use this one measured value as an estimate of the average value of many measurements. This is reasonably accurate if the one measurement is made over a sufficiently long time interval. (Why? Because over longer time intervals, N_0 is larger, so the fractional error $\sigma_{N_0}/N_0 = \sqrt{N_0}/N_0 = 1/\sqrt{N_0}$ is smaller. For example, if $N_0 = 100$, the estimated standard deviation is $\sqrt{100} = 10$, which is 10/100 = 10% error. If $N_0 = 10000$, the estimated standard deviation is $\sqrt{10000} = 100$, which is only 100/10000 = 1% error.)

To determine the uncertainty in δ and other calculated results, use the error propagation formula for a function f(x,y,…):

$$\sigma_f{}^2 = \left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2 + \cdots \tag{17}$$

If the polarizers were ideal, N(0°) would be $N_0/2$: exactly half the photon pairs would pass through the identically oriented polarizers. However, ε, the actual transmittance of light in the direction of the polarizer axis, is less than 1. When ε is accounted for, a factor of ε must be added to Eq. (3) for each of the two polarizers, yielding the corrected quantum mechanical prediction

$$\frac{N(\phi)}{N_0} = P_{actual}(\phi) = \frac{1}{2}\varepsilon^2 \cos^2\phi\,. \tag{18}$$

You can determine ε by setting ϕ = 0° and using measured values of N(0°) and N_0.

IN YOUR LAB REPORT

Theory (to include in lab report):

1. Quantum mechanical probabilities

Prove that Eq. (3) is ½$\cos^2\phi$. Hints:

Preliminary information: First let's think about just one photon. Suppose a photon is in a state

$$\frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right),$$

which is equivalent to 45° polarization. This photon has a 50% chance of passing through a horizontal polarizer. In general, what's the probability that a photon is transmitted through a horizontal polarizer? Answer: The probability is the square of the $|H\rangle$ coefficient, in our case, $\left(\frac{1}{\sqrt{2}}\right)^2$. Specifically, we perform the following steps:

- We multiply the state by $\langle H|$ and use the rules $\langle H|H\rangle = 1$ and $\langle H|V\rangle = 0$. These are exactly analogous to the unit vector rules $\hat{i} \bullet \hat{i} = 1$ and $\hat{i} \bullet \hat{j} = 0$.
- We compute the square of the absolute value of the result. In our 45° example, we obtain $\left|\langle H|\frac{1}{\sqrt{2}}\left(|V\rangle + |H\rangle\right)\right|^2$. Can you see why this is 50%?

Things are just a little more complicated when we have two particles to keep track of. The rules to follow are

$$\langle H|_A|H\rangle_A = 1,\ \langle V|_A|V\rangle_A = 1, \langle H|_B|H\rangle_B = 1, \langle V|_B|V\rangle_B = 1,$$

$$\langle H|_A|V\rangle_A = 0, \langle V|_A|H\rangle_A = 0, \langle H|_B|V\rangle_B = 0, \langle V|_B|H\rangle_B = 0$$

A symbol with an A subscript acts as a coefficient multiplying a symbol with a B subscript, and vice versa. So, for example, $\left(\langle V|_A\langle H|_B\right)\left(|H\rangle_A|H\rangle_B\right) = \langle V|_A|H\rangle_A\langle H|_B|H\rangle_A = 0\times 1 = 0$.

2. Freedman's Bell inequality

Fill in the steps omitted in the derivation:

- Prove the lemma, Eq. (6).
  - To derive the upper limit on U, look at two cases. When $x_1 \geq x_2$, factor $U = (x_1 - X)y_1 + (y_1 - Y)x_2 + (x_2 - x_1)y_2$. When $x_1 < x_2$, factor $U = x_1(y_1 - y_2) + (x_2 - X)y_1 - x_2(Y - y_2) \leq x_1(y_1 - y_2) + (x_2 - X)y_1 - x_1(Y - y_2)$. In each case, prove that the right hand side must be nonpositive.
  - To demonstrate the lower limit on *U*, factor *U*+*XY* three ways:

$$U + XY = (X - x_2)(Y - y_1) + x_1 y_1 + (x_2 - x_1)y_2 \text{ when } x_2 \geq x_1$$

$$U + XY = (X - x_2)(Y - y_1) + x_2 y_2 + x_1(y_1 - y_2) \text{ when } y_1 \geq y_2$$

$$U + XY = (X - x_2)(Y - y_1) + x_2 y_1 + (x_2 - x_1)(y_2 - y_1) \text{ otherwise}$$

- Derive Eq. (16) from Eq. (14). Replace every P with N by using Eq. (7) on each term. Then use $\phi = 22.5°$ to obtain one inequality. Obtain a second inequality by setting $\phi = 67.5°$, and use the fact that $P(\phi) = P(\phi{+}180°)$ to eliminate P(202.5°). Subtract one inequality from the other.

3. Quantum mechanical prediction for δ

Use Eqs. (16) and (18) to derive the quantum mechanical prediction for δ.

4. Error analysis

- Derive the formula for σ_δ in terms of the three measured quantities used to calculate δ.
- Derive the formula for the uncertainty in $N(\phi)/N_0$. This is the half-length of each error bar when you plot $N(\phi)/N_0$ vs. ϕ.
- Derive the formula for the uncertainty in ε.
- Derive for the formula for the uncertainty in the quantum mechanical prediction of δ, due to the uncertainty in ε.

Experiment (to include in lab report):

What is your final result, $\delta \pm \sigma_\delta$? Does it agree with the quantum mechanical prediction? Did you obtain a Bell inequality violation? What is the significance of this result?

Plot $N(\phi)/N_0$ vs. ϕ, preferably showing error bars. For comparison with theory, plot the right hand side of Eq. (18) on the same axes.

CLEAN-UP (actually please leave everything set up unless I ask you to put it away)

Beta barium borate is hygroscopic (it absorbs moisture). To keep it from fogging up, put the crystal pair in the jar of desiccant when you complete your measurements. Please be careful not to let the crystal pair (the small circle) touch anything! Dr. Seuss describes best the way I felt the first time I disassembled the apparatus:

Suppose, just suppose, you were poor Herbie Hart,
who has taken his Throm-dim-bu-lator apart!
He never will get it together, I'm sure.
He never will know if the Gick or the Goor
fits into the Skrux or the Snux or the Snoor.
Yes, Duckie, you're lucky you're not Herbie Hart
who has taken his Throm-dim-bu-lator apart!

--Dr. Seuss