Minimal course in fluid dynamics

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A minimal introduction to fluid dynamics, intended as useful background for soft matter scientists.

I. INTRODUCTION

Fluid flow includes problems such as the flow of our atmosphere, ocean currents, flow around airplanes and submarines, flow through pipes, and details of swimming bacteria. Phenomena studied include turbulence, chaotic mixing, and shock waves. Soft matter research tends to be concerned with slow flows and small scales, which allows for useful simplifications. This chapter mainly treats fluid dynamics in the slow and small limits, emphasizing concepts needed for a variety of soft condensed matter problems.

II. BASICS

A. Viscosity

"Fluid" is a general term that includes both liquids and gases. Fluids deform and flow under any applied force no matter how small. A flowing fluid does have dissipation as one parcel of fluid slides past another, and this is due to *viscosity*. At the atomic scale, this is the interaction of molecules of fluid as they move past one another, collide, and exchange momentum. At the macroscopic scale, this means that it requires force to cause a flow, as illustrated in Fig. 1. For a fluid confined between two plates of area A separated by distance d, the force required to move one plate at velocity v relative to the other plate is given by:

$$\frac{F}{A} = \eta \frac{v}{d} \tag{1}$$

with the viscosity η showing up as the proportionality constant. The equation is written such that the left side is stress (force per area) and the right side has the strain rate (v/d). η is a material parameter which has non-intuitive units of Pa·s=kg/m·s. For reference, water has $\eta = 1.00$ mPa·s and air has $\eta = 1.8 \times 10^{-5}$ mPa·s, both at 20° C. Pure glycerol has a viscosity 1500 times that of water, and honey is roughly 10^4 times more viscous than water. A longer discussion of viscosity is given in Sec. IV.
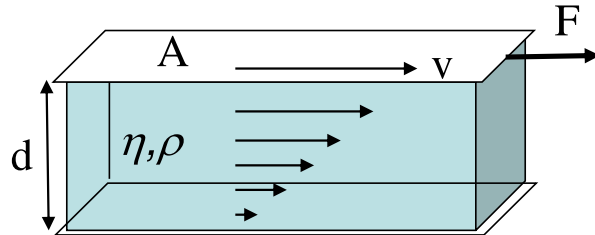
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FIG. 1. A fluid feeling stress F/A leading to strain rate v/d. F is an imposed force on the top plate which has area A, which moves at velocity v relative to the bottom plate a distance d below. The constant of proportionality between stress and strain is the viscosity η, with $F/A = \eta(v/d)$. η is a material property of the fluid, similar to the density ρ.

B. Coordinates

There are two complementary approaches to describing fluid motion, known as the Eulerian description and the Lagrangian description. The Eulerian description considers fixed positions in space and asks, at any moment in time, what the velocity of the fluid is at that position, $\vec{u}(\vec{r}, t)$ [1]. If the velocity is nonzero, then it is clear that the molecules composing the fluid at that position are changing with time. The Lagrangian approach takes the complementary viewpoint, of following a parcel of fluid as it moves in the flow. This is a bit problematic for a parcel of fluid which gets stretched and folded over time. On the other hand, if one observes the motion of a flowing fluid by following the trajectories of solid tracer particles in that fluid, then those tracer particles are most easiest thought of in a Lagrangian sense.

The Eulerian approach is more common, and in practice one does not usually worry about the molecular nature of the fluid but rather treats the fluid as a continuum. (This is a reasonable approximation as long as you are concerned with length scales one or two orders of magnitude larger than the inter-molecular spacing, which is almost always the case.)

C. The Navier-Stokes equation and continuity equation

The starting point for fluid mechanics is Newton's second law, $\vec{F} = m\vec{a}$. Let's modify it to be more useful for fluids. The density of a fluid is not required to be constant, so it's better to consider the momentum form

of this equation: $\vec{F} = d\vec{p}/dt$. Rather than considering point masses, it's easier to think about momentum density and body forces per unit volume or per unit mass. (Body forces are forces such as gravity that typically originate outside the fluid.) Additionally the adjacent fluid exerts stresses that can change the momentum density. The resulting equation is known as the Cauchy Equation, a general equation for momentum conservation, which we will skip over for the sake of brevity.

The Navier-Stokes equation is a specific version of the Cauchy equation. In particular, the Navier-Stokes equation recognizes that the pressure and viscosity play important roles for fluids. For a formal derivation the reader can consult a textbook (for example Ref. [2]). Here, we will state the equation and then explain the meaning of the various terms. The momentum density of a fluid evolves as:

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} \right] = -\rho \nabla P + \eta \nabla^2 \vec{u} + \vec{g}. \qquad (2)$$

The left side of this equation is $d\vec{p}/dt$, the change of momentum density at a given position. The first term, $\partial \vec{u}/\partial t$, represents that the velocity can be changing at that location thus changing the momentum. The second term is trickier. The momentum of a fluid parcel can change by moving to a location where the velocity is different. $\vec{u}$ is the velocity and thus indicates where the fluid parcel is moving; $\vec{u} \cdot \nabla$ then asks for the spatial derivative in that direction; and finally $(\vec{u} \cdot \vec{\nabla})\vec{u}$ yields the spatial change of the velocity in that direction. This term is referred to as the inertial term, and could be thought of as the inertia of a fluid parcel ($\rho\vec{u}$) carrying that fluid parcel to a new region where the momentum is different.

More generally, one defines the convective derivative:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\vec{u} \cdot \nabla)f \qquad (3)$$

with the idea being the same. Quantity f can change by changing at that location, or by being carried by the fluid to a new location where f is different. When applied to momentum, the left side of the Navier-Stokes equation (Eqn. 2) can be written as

$$\rho\frac{D\vec{u}}{Dt} = \rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} \right]. \qquad (4)$$

The right-hand side of Eqn. 2 has three terms, which are the pressure term, viscous dissipation term, and body force term. The pressure term is straightforward: if pressure is uniform in space, it has no effect on fluid flow. If the pressure is large at point A and small at B, then it pushes the fluid from A to B (which is why the negative sign is in front of $\vec{\nabla}P$, so that the fluid accelerates down the pressure gradient). The viscous term depends on two spatial derivatives of the velocity field. The argument presented in Sec. II A is that a spatial variation in the velocity should result in a force, so one might expect the viscous term to look like $\vec{\nabla} \cdot \vec{u}$. However, if there

is a uniform gradient in the velocity field, then the viscous forces acting on either side of a parcel of fluid are equal and opposite in direction, and thus do not change the momentum of that fluid parcel. The gradient cannot be uniform, and working through the relevant math one finds that the correct second order spatial derivative is $\nabla^2 \vec{u}$ [2]. The third term on the right hand side of Eqn. 2, $\vec{g}$, represents any body forces. These forces can be gravitational, or perhaps electrical or magnetic if the fluid has some special composition and the environment has the appropriate fields.

Equation 2 is for momentum conservation. We also need mass conservation, which is given by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{u}) = 0. \qquad (5)$$

If the density is decreasing at some point (first term negative), then this is because the mass is diverging from that point (second term positive). In other words, the second term is the mass flux away from a given position. Equation 5 is known as the continuity equation.

Many fluids are incompressible, or can be treated as such. For example, water has a bulk modulus of about $K = 2.2$ GPa which is only about a factor of 20 smaller than glass. A 1% change in volume requires a pressure of $0.01K \approx 220$ atm, a pressure unlikely to be encountered in soft matter science. Accordingly, we can take the density ρ to be a constant, and simplify the continuity equation to be

$$\vec{\nabla} \cdot \vec{u} = 0. \qquad (6)$$

Note that for gases the incompressibility assumption is usually false. A bubble rising up through water will change its volume by 1% for every 10 cm of rise, due to the depth-dependence of the hydrostatic pressure of the water.

D. Reynolds number

The most famous nondimensional number in fluid mechanics is the Reynolds number, and for soft condensed matter science it is famously small. We can derive this number by asking about the relative importance of two specific terms in the Navier-Stokes equation, the inertial term ($\rho(\vec{u} \cdot \vec{\nabla})\vec{u}$) and the viscous term ($\eta\nabla^2\vec{u}$). Of course, $\vec{u}$ is in general space and time dependent, so the magnitude of these two terms will vary quite a bit. However, we can do a useful estimate. Define U as the characteristic velocity scale, and L as the characteristic length scale over which U changes. In other words, $\vec{\nabla}\vec{u} \sim U/L$ will give us the correct order of magnitude. Then define the Reynolds number as

$$Re = \frac{\text{inertial term}}{\text{viscous term}} = \frac{\rho(\vec{u} \cdot \vec{\nabla})\vec{u}}{\eta\nabla^2\vec{u}} \approx \frac{\rho U^2/L}{\eta U/L^2} = \frac{\rho UL}{\eta}. \quad (7)$$

Re is nondimensional, given that it was constructed by taking the ratio of two terms which have the same dimensionality. It is easy to see that Re will be large for fast flows (airplanes) or large flows (oceans, atmospheres) or both (Jupiter's Great Red Spot). As a simple example, consider sitting in a car moving at 100 km/hr and sticking your hand out the window. $U = 100$ km/hr= 28 m/s, $L \approx 0.1$ m for your hand, $\rho_{air} \approx 1.2$ kg/m^3, and $\eta_{air} = 1.8 \times 10^{-5}$ mPa·s, giving Re$\approx 2 \times 10^5$. The meaning of this is that the velocity field describing the air flow around your hand is not greatly influenced by viscosity. The force you feel from the air on your hand is not a viscous force, but rather the force required to accelerate the air up to the velocity of your hand. Mathematically, if you wished to solve the Navier-Stokes equation to describe the air flow around your hand, you would neglect the viscous term in that equation. This is somewhat of a simplification, in that very close to your skin viscosity *does* play a role in the "viscous boundary layer;" this is described in detail in Ref. [2].

The complementary case holds for small materials and situations. The reader should immediately put down this chapter and go read the classic article "Life at Low Reynolds Number" by E. M. Purcell [3] for a beautiful explanation of low Reynolds number physics. To give one example distinct from Purcell's article, consider a 10 μm diameter polystyrene particle sedimenting in water. The sedimentation velocity is $U \approx 10$ μm/s (see Sec. III C), and so Re$= 10^{-4}$. Now, the conceptual meaning is that inertia no longer plays a role, and the $(\vec{u} \cdot \vec{\nabla})\vec{u}$ term can be neglected in Eqn. 2. As this is the only term that is nonlinear in $\vec{u}$, this is a quite useful simplification.

The Reynolds number can be derived in a different fashion by non-dimensionalizing the Navier-Stokes equation; see Ref. [2] for this derivation.

E. Choosing U and L

Sometimes the choice of U and L is not obvious. For U one typically chooses the largest velocity present, most often the velocity of a boundary to the flow. For L, the idea is that we are assuming $\vec{\nabla} \sim 1/L$ so one should look for the length scale over which the velocity changes. For example, Fig. 2 shows a Taylor-Couette flow cell, one of the canonical geometries for fluid studies, in particular of studies of turbulence. Here, the velocity scale would be ωR_1 and the length scale over which the velocity varies is $R_2 - R_1$. Thus, Re $= \rho \omega R_1 (R_2 - R_1)/\eta$. Despite the prominence of the symbol L in the diagram, it would not be the right length scale for determining Re.

F. Stokes Equation

The Stokes Equation is the simplification of the Navier-Stokes Equation (Eqn. 2) for Re $\ll 1$, in other words,
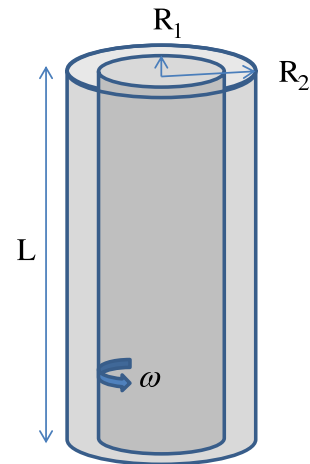

FIG. 2. Sketch of a Taylor-Couette apparatus.

dropping out the inertial term. An additional simplification is to make the equation time-independent. Purcell has a nice argument explaining this [3]: at low Reynolds number, viscosity is very significant, so that if a force driving a flow is suddenly removed, the flow will stop nearly instantaneously. In other words, the fluid flow is determined at any moment by the boundary conditions, external forces, and pressure – but not by any history. Thus, the Stokes equation is

$$-\rho \nabla P + \eta \nabla^2 \vec{u} + \vec{g} = 0. \qquad (8)$$

G. Boundary conditions

At the boundary between a solid and a fluid, the fluid obeys a "no-slip" boundary condition. This means that the fluid velocity must match the velocity of the solid. For example, for a fluid flowing through a pipe, the fluid velocity must be zero at the pipe walls. Note that the matching of velocities is true in a vectorial sense: the velocity must be the same for both normal and tangential velocities.

At the boundary between two fluids (say, oil and water), the fluids obey a "stress-free" boundary condition. At these boundaries, the velocity of the fluids are equal at the interface (again, true for both normal and tangential components of the velocity). It is certainly possible for these components to be nonzero. If the normal component is nonzero, then the interface is moving. Examples include the air-water interface of ocean waves, or the oil-water interface for an oil droplet moving upward in water. In addition, the stress (related to derivatives of the velocity) relates to the interface motion, surface tension, and viscosities of the two fluids. I'd explain more, but I'm writing this on January 8, 2017 and plan to hand these notes out on January 9, so I'll note that the stress-free boundary condition is described in more detail online in various places, and also in any fluids textbook.

III. SPHERES

A. Drag force on moving sphere

Physicists are known for approximating complex objects as spheres. Indeed, for many situations the sphere is the simplest object to consider, and that is true for fluid mechanics. That being said, spheres are of particular interest for soft matter as many constituents of soft materials are spherical. Colloidal particles are easiest to make as spheres, although more complex shapes are possible [4]. Emulsion droplets prefer to be spherical due to surface tension. Likewise, bubbles in a foam prefer a spherical shape due to their surface tension.

Furthermore, as it happens, the drag force on particles at low Reynolds number doesn't vary that much with shape. By dimensional analysis, the drag force on a moving object should be $F \sim \eta L v$ based on the viscosity η, a characteristic length L, and the velocity v. For spheres, the exact result is due to Stokes and is

$$F_{\text{spheres}} = 6\pi\eta a v \tag{9}$$

using the sphere radius a for the characteristic length scale. For a long ellipsoidal needle moving parallel to its long axis, the drag force increases to

$$F_{\text{needle,parallel}} = \frac{2\pi\eta L v}{\ln(L/a)} \tag{10}$$

in terms of the needle length L and radius a. If $L/a = 100$ then this is 7.2 times larger than the drag on a sphere. For the same needle moving perpendicular to its long axis, the results are

$$F_{\text{needle,perpendicular}} = 2F_{\text{needle,parallel}}. \tag{11}$$

Approximating this needle as a sphere of the same volume would suggest a drag force 4.6 times larger, so too small by a factor of ~ 3, but at least correct to within an order of magnitude. The drag force for a variety of shapes is compiled in Ref. [5].

For droplets of one fluid moving through another fluid, stresses at the surface of the droplet will induce flow inside the droplet, which reduces the drag force acting on the droplet. Again restricted to low Re, the drag force on a droplet is

$$F_{\text{droplet}} = \Big[\frac{1 + (2\eta_{\text{bulk}}/3\eta_{\text{drop}})}{1 + (\eta_{\text{bulk}}/\eta_{\text{drop}})}\Big]F_{\text{sphere}}, \tag{12}$$

where η_{bulk} is the background fluid viscosity (which is what is used for F_{sphere} in this equation) and η_{drop} is the viscosity of the droplet fluid. The term in square brackets approaches 2/3 for $\eta_{\text{drop}} \ll \eta_{\text{bulk}}$ and approaches 1 for $\eta_{\text{drop}} \gg \eta_{\text{bulk}}$. The latter limit can be understood as the limit in which the droplet viscosity is so high that it acts like a solid particle, and thus must simplify to Eqn. 9. The former limit is correct to within 1% for

small air bubbles rising in water at room temperature. (All of these drag force results are correct to within 1% for Re $\lesssim 10^{-2}$, so this requires that small air bubble to be about 15 μm radius or smaller.)

B. Flow field around moving sphere

Related to the drag force is the flow field around a sphere moving with a specified velocity (still at low Re). This can be found by solving the Stokes Equation in the reference frame where the sphere is at rest and the background fluid has velocity v, and using the boundary condition that the velocity is zero at the sphere surface; see Ref. [2] for a derivation. The derivation and solution are best done in spherical coordinates, with the result being:

$$U_r = U_\infty \cos\theta\Big[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\Big] \tag{13}$$

$$U_\theta = -U_\infty \sin\theta\Big[1 - \frac{3a}{4r} + \frac{a^3}{4r^3}\Big]. \tag{14}$$

In the limit $r \to \infty$, the flow field reduces to U_∞ in the direction $\theta = 0$. In these equations r is the distance from the center of the sphere. The drag force can be calculated from the viscous stresses exerted by the flow field on the surface of the sphere [2].

Changing back into the reference frame where the sphere has velocity U_∞ and the surrounding fluid is motionless far from the sphere, the key result is that the flow field decreases toward zero as $1/r$. This means that a moving sphere exerts a hydrodynamic force on a nearby sphere with the magnitude of this force inversely proportional to their separation. For spheres undergoing Brownian motion, this results in measurable correlations in the motion of pairs of particles [6]. This correlation decays as $1/r$. Furthermore, the direction dependence of Eqns. 14 means that the correlations are twice as strong for displacements of spheres along the direction of their separation as compared to displacements of those spheres perpendicular to the direction of their separation. Intriguingly, the strain field around a displaced particle in a perfectly elastic material also decays as $1/r$ [7]. This coincidence leads one to note that the correlations between particle displacements will decay as $1/r$ for any homogeneous viscoelastic medium [6, 8] which has been experimentally observed [6].

A final point to note is that these results for the drag force and flow fields are for objects moving in an infinitely large fluid system. However, often in soft matter one encounters boundaries – for example, solid walls in a microfluidic device, or a second immiscible liquid. Particles moving near a solid flat wall experience an increased drag force, due to the need to shear the fluid between the particle and the wall; this is known as Faxén's law [9, 10]. This is true for particles moving either parallel or perpendicular to the wall. If particles move near a flat boundary between two liquids, then the drag force depends both on the viscosity of the liquid they are in,
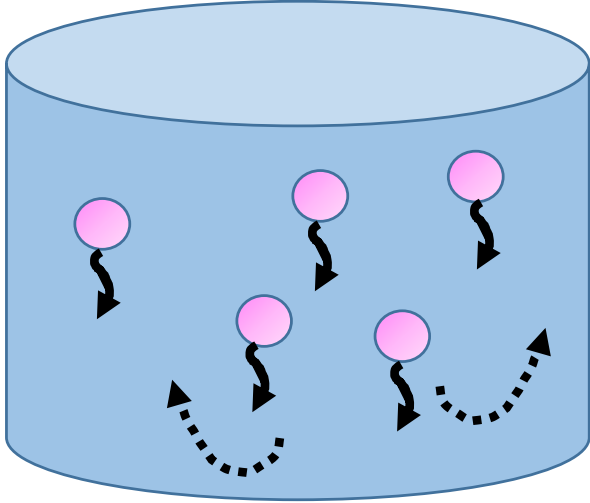
FIG. 3. Sketch of sedimenting particles in a container. The wavy solid arrows indicate that particle trajectories can be erratic [14]; each particle sets up a flow field around it roughly like Eqns. 14 and the interacting flow fields cause the erratic motions. The $1/r$ flow field around each particle also results in nontrivial interactions with the container boundaries, along the lines of Faxen's law. Additionally, there is a fluid backflow (indicated by the dashed lines) as the particles approach the bottom of the container.

and also the viscosity of the second nearby liquid. In this case if that second liquid has a lower viscosity, the drag force on the moving particle can be reduced [11–13]. If the second liquid has a significantly larger viscosity, then the results approach the limit for a hard wall, that is, the results approach Faxén's law.

C. Sedimentation

A common situation is sedimentation: for example, dense particles sinking through a less dense fluid. For a single particle, the situation is easy. The gravitational force is given by the buoyant weight:

$$F_{\text{grav}} = \Delta\rho(\frac{4}{3}\pi a^3)g \tag{15}$$

where $\Delta\rho$ is the density difference between the particle and the fluid, the term in parentheses is the particle volume, and g is the acceleration due to gravity. The gravitational force is balanced by the drag force (Eqn. 9) leading to the sedimentation velocity

$$v_{\text{sed}} = \frac{2}{9}\frac{\Delta\rho g a^2}{\eta}. \tag{16}$$

This was used above in Sec. III, along with Eqn. 7, to estimate the size of an air bubble that has Re$\sim 10^{-2}$.

However, this simple result for one particle is not correct for a sedimenting collection of particles due to their hydrodynamic interactions; see Fig. 3. Furthermore, in any real experiment with container boundaries, sedimenting particles are slowed by the backflow of fluid that has to move past them. In practice the sedimentation velocity of a suspension of spheres is complicated and depends on the container size, container side walls, particle concentration, and Reynolds number [14–18].

IV. VISCOSITY

A fluid's viscosity is clearly an important parameter, showing up for example in the Reynolds number (Eqn. 7) and the drag force (Eqn. 9). Accordingly, this section provides additional information about fluid viscosity.

The first important point is that a fluid's viscosity is a strong function of temperature. For room temperature water, η changes by about 2% per degree Celsius. Thus, if you really care what viscosity a fluid has in an experiment, you must also take care to measure the temperature (and preferably control it). For example, you can measure the size of small colloidal particles by measuring their diffusivity, but this depends on knowing the viscosity – so a 1° uncertainty in temperature is at least a 2% uncertainty in particle size [19].

Colloids are materials made of small solid particles in a liquid [20]. Common examples are blood, paint, and toothpaste. Often one wishes to know the viscosity of a colloidal sample, and it is well known to depend on the fraction of volume occupied by the solid particles – that is, the volume fraction ϕ. For $\phi \ll 1$, the viscosity is known to be

$$\eta(\phi)/\eta_{\text{fluid}} = 1 + \frac{5}{2}\phi + 6.2\phi^2, \tag{17}$$

with the first order correction due to Einstein [21] and the second order correction due to Batchelor [22]. For volume fractions $\phi \gtrsim 0.1$ there are several different fitting functions in use [23, 24]. In particular, colloids with $\phi \approx 0.6$ are extremely viscous, about 3 - 4 orders of magnitude more viscous than η_{fluid} [24]. At this point, it is quite challenging to measure the viscosity accurately [23]. In fact the effective viscosity can depend strongly on the flow rate: initially as the flow rate increases, the apparent viscosity decreases (termed "shear-thinning") and then at higher flow rates the apparent viscosity increases (termed "shear-thickening") [25]. Measuring the flow properties of dense colloidal fluids is an active research field and the interested reader should consult other sources [20, 24, 25]; the main point for this chapter is that the viscosity of colloids is simple at low volume fractions (Eqn. 17) and complicated at higher volume fractions. In all cases, the viscosity of a colloid is scaled by the viscosity of the background fluid η_{fluid}.

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