requires equality between systems of both the Reynolds number and the Mach number, $Ma = U/a$ (a is the speed of sound; see Section 5.8). Thus

$$C_D = f(\text{Re}, \text{Ma}). \tag{7.26}$$

4. When the flow is unsteady as a result of changes in the imposed conditions, these changes will have a time scale Ψ associated with them. In problems such as the above there is then the additional non-dimensional parameter $U\Psi/L$, and dynamical similarity throughout the development of the flow requires equality of this in addition to the Reynolds number.

It should be noted that, in the context of model testing, the above discussion of dynamical similarity is the statement of an ideal. It is often not possible in practice to make all the governing non-dimensional parameters the same as on the full scale. In ship model testing, for instance, a reduction in L requires an increase in U to keep the Reynolds number the same but a reduction in U to keep the Froude number the same (since there is little manoeuvreability of ρ, ν, and g). Hence, tests have to made without full dynamical similarity, and special attention must be given to the errors arising in the transfer of information to the full scale.

8
LOW AND HIGH REYNOLDS NUMBERS

8.1 Physical significance of the Reynolds number

The Reynolds number, introduced in the last chapter in the context of dynamical similarity, can be given a physical interpretation. This is useful in gaining an understanding of the dynamical processes that are important in different Reynolds number ranges, and in formulating corresponding approximations to the equations of motion.

To discuss this we need a name for each of the terms in the dynamical equation of steady incompressible flow:

$$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \mu \nabla^2 \boldsymbol{u}. \tag{8.1}$$

The second and third terms are given the obvious names pressure force and viscous force. The first term is called the inertia force. Physically, it is not a force, but it has the dimensions of force per unit volume and it is sometimes convenient to think of the dynamical equation in terms of a static balance between forces. The procedure is analogous to the more familiar use of the term centrifugal force to represent the acceleration involved in circular motion. No new idea is involved here, just a new name.

In the non-dimensional form of eqn (8.1),

$$\boldsymbol{u}' \cdot \nabla' \boldsymbol{u}' = -\nabla'(\Delta p)' + \frac{1}{\text{Re}} \nabla'^2 \boldsymbol{u}' \tag{8.2}$$

(cf. eqn (7.13)), the primed quantities (possibly excepting $(\Delta p)'$) may be expected to be of order unity in magnitude. We shall see later that there are important qualifications to that statement. However, as a starting point it is justified so long as the length and velocity scales, U and L, have been chosen as typical quantities. Then a general speed will be of order U and $|\boldsymbol{u}'| \sim 1$; a general distance over which quantities vary significantly will be of order L and $\partial/\partial x'$, etc. will be of order unity.

Hence the ratio of the first term to the third in eqn (8.2) is of order Re. The corresponding terms in eqn (8.1) are in the same ratio. This indicates a physical interpretation of the Reynolds number as

$$\text{Re} \sim \frac{\text{inertia forces}}{\text{viscous forces}}. \tag{8.3}$$

An alternative (entirely equivalent) formulation of this result, cited because we shall proceed in this way in subsequent chapters, is to write

$$|\boldsymbol{u}\cdot\nabla\boldsymbol{u}| \sim U^2/L, \quad |\nu\nabla^2\boldsymbol{u}| \sim \nu U/L^2. \tag{8.4}$$

Hence

$$\frac{|\boldsymbol{u}\cdot\nabla\boldsymbol{u}|}{|\nu\nabla^2\boldsymbol{u}|} \sim \frac{UL}{\nu} = \mathrm{Re}. \tag{8.5}$$

The Reynolds number thus indicates the relative importance of two dynamical processes. At a general point within the flow, the ratios of these two terms will not be exactly equal to the Reynolds number, but their characteristic magnitudes will be in this ratio.

8.2 Low Reynolds number

When the Reynolds number is much smaller than unity the viscous force dominates over the inertia force so much that the latter plays a negligible role in the flow dynamics. One may use an approximate form of the equation of motion with the inertia term dropped. Equation (8.2) becomes

$$0 = -\nabla'(\Delta p') + \frac{1}{\mathrm{Re}}\nabla'^2\boldsymbol{u}', \tag{8.6}$$

these terms being of order 1/Re and the neglected term of order 1. The pressure term must be retained since it is necessary to match the number of variables to the number of equations (Section 5.6). Physically, this means that the size of the pressure term is always governed by the other dynamically important terms—in this case by the viscous term.

Reverting to the dimensional form, eqn (8.6) is

$$\nabla p = \mu\nabla^2\boldsymbol{u}. \tag{8.7}$$

At every point in the fluid there is an effective balance between the local pressure and viscous forces. Equation (8.7) is known as the equation of creeping motion. It is evidently much simpler than the full Navier–Stokes equation, and solutions have been found for many cases for which the full equation has not yielded a solution. One case will be discussed in Section 9.4. Such solutions are found to agree well with the observed behaviour at low Reynolds number (see, e.g., Fig. 9.3), thus justifying the procedure leading to the approximation.

Two characteristic features of low Reynolds number flow are worth mentioning. Firstly, solutions of the equation of creeping motion are reversible; that is to say, if one has a solution, then there is another one

with the same streamline pattern but with the flow everywhere in the opposite direction (with all pressure gradients reversed). Hence, for example, the flow from right to left past an obstacle is the exact reverse of that from left to right. By extension, if one has an obstacle of a shape having upstream–downstream symmetry (its rear half is the mirror image of its front half), then the whole flow pattern has this symmetry; the pressure distribution is antisymmetric. We shall not derive these results formally, but it is readily seen that the solution to be presented in Section 9.4 possesses the above properties. We shall also be seeing some flows with this symmetry in Figs. 12.1, 12.6, and 12.7.

The second characteristic feature of low Reynolds number flows is that viscous interactions extend over large distances. For example, particles sedimenting at low Reynolds number affect each other's motion even when their separation is large compared with their size. Figure 8.1 illustrates this long-range viscous action for flow past a circular cylinder. It re-presents the information of Fig. 3.2 to show the velocity distribution across the mid-plane at a Reynolds number of 0.1. In the next section we shall be looking at the corresponding figure for high Reynolds number flow; comparison of the two provides a good illustration of the way in which different dynamical processes dominate in different Reynolds number ranges.

In this book further discussion of low Reynolds number flows will be confined to Sections 9.4 and 9.5 and parts of Chapter 12. There are

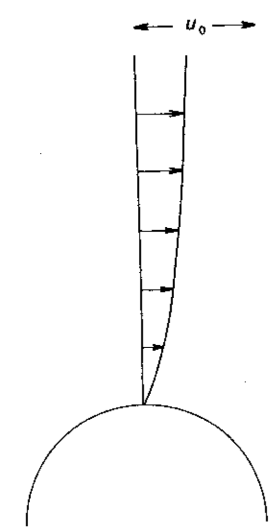

FIG. 8.1 Velocity distribution on centre plane in flow past circular cylinder at Re = 0.1.

nevertheless many interesting low Reynolds number phenomena, as is particularly well illustrated by the film of Ref. [53].

8.3 High Reynolds number

Corresponding arguments for high Reynolds number flow indicate that the viscous force is so small compared with the inertia force that it can be neglected. Equation (8.2) then approximates to

$$\boldsymbol{u}' \cdot \nabla \boldsymbol{u}' = -\nabla'(\Delta p)' \tag{8.8}$$

or dimensionally

$$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p. \tag{8.9}$$

This is Euler's equation of inviscid motion. When it applies, the fluid at each point has an acceleration directly related to the pressure gradient.

The argument applies also to unsteady flow ($\mu \nabla^2 \boldsymbol{u}$ being negligible compared with $\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ regardless of the size of $\rho\, \partial \boldsymbol{u}/\partial t$), and a more general form of Euler's equation is

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p. \tag{8.10}$$

The relationship of this equation to the actual behaviour at high Reynolds numbers is much more complex than the relationship of the creeping motion equation to low Reynolds number flow. Comparing Euler's equation with the Navier–Stokes equation, we see that the discarded term is the highest-order differential term: the only one involving second space derivatives. The approximation thus reduces the order of the differential equation. A corresponding reduction must be made in the number of boundary conditions.

We saw in Section 5.7 that the no-slip boundary condition is a consequence of the action of viscosity. One may thus expect that this is the condition that should be discarded for mathematical consistency with the inviscid equation. This is indeed the case; we shall see in Chapter 10 that solutions of Euler's equation are obtained by matching only to the impermeability condition (eqn (5.33)). Imposition of the no-slip condition also would result in no solution being obtainable.

We now have a paradoxical situation. The statement that Euler's equation applies at high Reynolds number means, more precisely, that as the Reynolds number is increased the viscous term becomes relatively smaller and smaller, although never absolutely zero; Euler's equation becomes a better and better approximation. A boundary condition, however, cannot be similarly relaxed as the Reynolds number increases.

It either applies or it does not apply; there is no meaning to the statement that the no-slip condition is present but to a negligible extent. On the other hand, one would not expect there to be some Reynolds number at which the no-slip condition suddenly 'switches off', and it is found experimentally that it continues to apply no matter how high the Reynolds number.

Consequently, the viscous term in the dynamical equation must always remain significant in the vicinity of a boundary, so that the equation remains of the order appropriate to the boundary conditions. The region in which this happens is known as the boundary layer. The reasoning (eqns (8.2)–(8.5)) that the viscous force should be negligible breaks down in the boundary layer because the flow develops an internal length scale much smaller than the imposed length scale, L. This is the boundary layer thickness, δ. We shall see below, and in more detail in Chapter 11, that the size of the viscous term can be determined by δ whilst the size of the inertia term is still determined by L:

$$|\boldsymbol{u} \cdot \nabla \boldsymbol{u}| \sim U^2/L, \quad |\nu \nabla^2 \boldsymbol{u}| \sim \nu U/\delta^2. \tag{8.11}$$

Inertia and viscous forces can thus remain of comparable order of magnitude if

$$U^2/L \sim \nu U/\delta^2 \tag{8.12}$$

that is if

$$\frac{\delta}{L} \sim \left(\frac{UL}{\nu}\right)^{-1/2} = \mathrm{Re}^{-1/2}. \tag{8.13}$$

The difference between the two length scales must become more marked as the Reynolds number increases.

We consider first the simplest example of a boundary layer. Suppose that a flat plate of negligible thickness is placed in a uniform stream, speed u_0, parallel to it (Fig. 8.2). Then, for a theoretical situation governed by Euler's equation and without the no-slip condition, the flow could obviously continue as if the plate were not there (Fig. 8.2(a)). The speed would be u_0 everywhere. In a real fluid, at large Reynolds number, the speed remains very close to u_0 over most of the flow. Close to and behind the plate, however, there are regions in which a large change occurs (Fig. 8.2(b)). These are the boundary layers on either side of the plate and the wake behind it. We shall not consider the wake further at the moment, but it can be thought of as the extension of the boundary layers downstream.

The rapid variation of speed in the boundary layer gives rise to much larger values of $\partial^2 u/\partial y^2$ than would otherwise exist. This makes the viscous force much larger and so makes it appropriate to use (8.11), rather than (8.4), for the orders of magnitude.

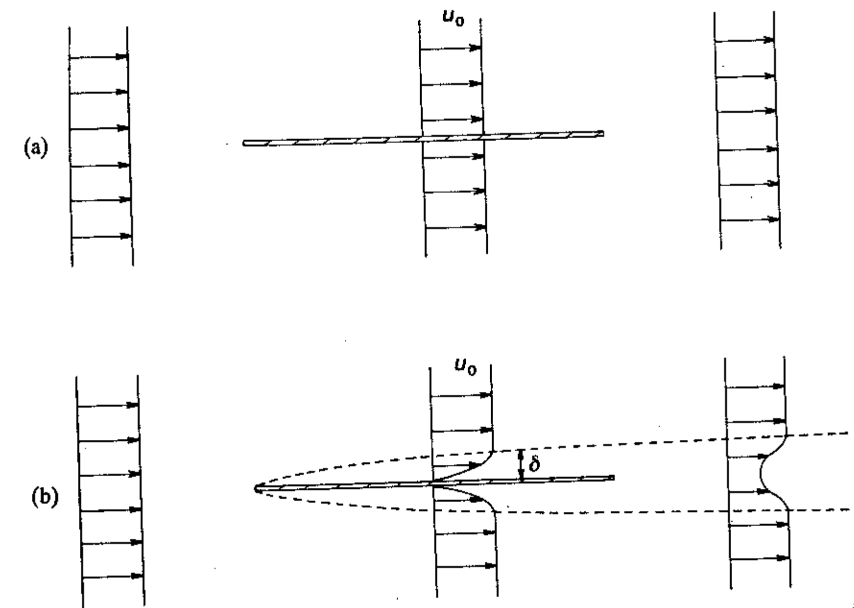

FIG. 8.2 Velocity profiles in flow past thin plate: (a) imagined inviscid flow; (b) real fluid at high Reynolds number. Dotted lines indicate edges of boundary layer and wake.

It is often useful to have a precise definition of the boundary layer thickness, δ (shown in Fig. 8.2(b)). There is, of course, no line beyond which the presence of the plate has absolutely no effect; the velocity still approaches asymptotically to u_0, but very rapidly. A common procedure is to choose δ such that

$$u = 0.99u_0 \quad \text{at} \quad y = \delta \tag{8.14}$$

(y is distance from plate); the boundary layer is taken to be the region in which the velocity differs by more than 1 per cent from the free-stream velocity.

The longitudinal length scale L is provided in this example by the distance from the leading edge. We shall see in Section 11.4 (eqn (11.32)) that

$$\frac{\delta}{x} \propto \left(\frac{u_0 x}{\nu}\right)^{-1/2}, \tag{8.15}$$

as expected from relationship (8.13).

The edge of the boundary layer is *not* a streamline. The only significance of the line, $y = \delta$, shown in Fig. 8.2(b) is that indicated by

eqn (8.14). Fluid crosses this line. In the present example, fluid just outside the boundary layer at one value of x is inside it at larger x.

For any obstacle other than a flat plate parallel to the free-stream, the situation is more complicated. The fluid is diverted past the obstacle and the solution of Euler's equation is not just a uniform flow (which would not satisfy the impermeability boundary condition). It would thus be meaningless to define the boundary layer and wake as the regions in which the velocity departs significantly from u_0. They are defined instead as the regions in which the action of viscosity significantly affects the velocity distribution. Suppose one has found a solution of Euler's equation for the flow past such an obstacle (which we may call the inviscid flow solution). This will not satisfy the no-slip condition. Close to the wall of the obstacle, viscous action will modify the flow so that the no-slip condition is obeyed. The region in which this happens is the boundary layer.

We can thus generalize the specification of a boundary layer implied by eqn (8.14): the boundary layer is the region in which the velocity differs by more than 1 per cent from the inviscid flow solution.

Figure 8.3 shows an example (cf. Fig. 8.1 for the corresponding example at low Reynolds number). The inviscid flow solution for flow past a circular cylinder gives a velocity profile across the mid-plane as shown by the broken curve. The speed at the wall is twice the free-stream

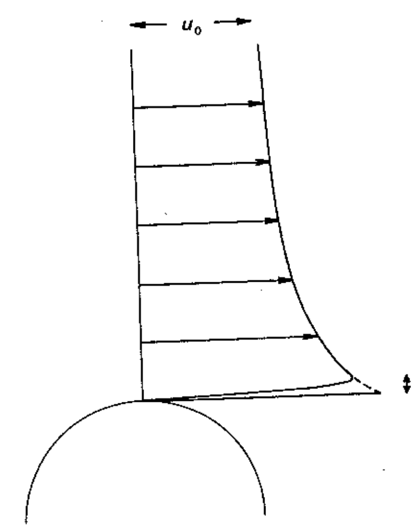

FIG. 8.3 Inviscid (broken line) and high Reynolds number (full line) velocity distributions across centre plane in flow past circular cylinder (neglecting effects of boundary layer separation).

speed. The full curve shows a high Reynolds number velocity profile, satisfying the no-slip condition. The boundary layer is the region in which the two profiles differ significantly.†

Chapter 10 will describe the elements of inviscid flow theory. Chapter 11 will discuss boundary layers. We can now see the role of these two aspects in the development of our understanding of flow at high Reynolds numbers. The first stage in tackling a new problem is usually a solution of the inviscid flow problem. This will, subject to qualifications made below, describe most of the flow. It does so, however, only because the region in which it applies is separated off from the no-slip condition by the boundary layer. One of the pieces of information obtained from the inviscid flow solution is the pressure distribution over the boundaries. This is part of the input to the second stage, a treatment of the boundary layer. If this indicates that the boundary layer undergoes separation (see Section 12.4), some modification to the inviscid solution will be required; in principle an iterative procedure is then needed, though in practice it may be difficult.

Historically, Euler's equation is older than the Navier–Stokes equation, and it was puzzling that it described some aspects of fluid behaviour very well whilst failing totally to describe others. We can now see the reason: those aspects described well were those unaffected by the presence of boundary layers. An important feature that does depend on boundary layers is the force on an obstacle, and we shall see that it was with this that failure of Euler's equation was particularly dramatic. The introduction of the concept of boundary layers, by Prandtl in 1904, was a landmark in the history of fluid dynamics; a very high proportion of subsequent developments stem directly from it.

We have seen in Chapters 2 and 3 that flow at high Reynolds number is prone to instability. Above we have been thinking mainly of laminar flow. The discussion is, however, by no means academic. Transition to turbulence occurs in regions such as boundary layers and wakes, whilst laminar flow continues in inviscid regions. Hence, the division into the two regions is still useful.

It is apparent that a much wider range of phenomena occur at high Reynolds number than at low. For this reason alone one would wish to give the former situation much more extensive consideration in a book of this sort. However, it is worth noting that this emphasis coincides to a large extent with the type of situation one meets most frequently in

† We know from Chapter 3 that Fig. 8.3 involves some oversimplification. In the first place, over a wide Reynolds number range, the separation of the flow from the wall leading to the formation of attached eddies occurs upstream of the station shown. Secondly, even when it occurs downstream, the existence of the separation will modify the inviscid flow solution—see Section 12.5.

practical situations. The values of the kinematic viscosity for water and air (at common values of the temperature and pressure) are respectively $1.0 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$ and $1.5 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$. In either fluid one needs only an object of a few centimetres in size moving at a speed of a few centimetres per second to reach a moderately high Reynolds number.