PHYSICAL FLUID DYNAMICS

Second Edition

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EQUATIONS OF MOTION

5.1 Introduction

We turn now to the formulation of the basic equations—the starting point of the theories that will lead, one hopes, to an understanding of phenomena such as those described in Chapters 2 to 4. These equations are formulations appropriate to a fluid in motion of the usual laws of mechanics—conservation of mass and Newton's laws of motion. In some situations other physical processes may be present, thermodynamic processes for instance, and equations for these are similarly formulated, as we shall examine more fully in Chapter 14.

5.2 Fluid particles and continuum mechanics

Before we can proceed with this formulation we need certain preliminary ideas, the most important being the concept of a fluid particle.

The equations concern physical and mechanical quantities, such as velocity, density, pressure, temperature, which will be supposed to vary continuously from point to point throughout the fluid. How do we define these quantities at a point? To do so we have to make what is known as the assumption of the applicability of continuum mechanics or the continuum hypothesis. We suppose that we can associate with any volume of fluid, no matter how small, those macroscopic properties that we associate with the fluid in bulk. We can then say that at each point there is a particle of fluid and that a large volume of fluid consists of a continuous aggregate of such particles, each having a certain velocity, temperature, etc.

Now we know that this assumption is not correct if we go right down to molecular scales. We have to consider why it is nonetheless plausible to formulate the equations on the basis of the continuum hypothesis. It is simplest to think of a gas, although the considerations for a liquid are very similar.

The various macroscopic properties are defined by averaging over a large number of moelcules. Consider velocity for example. The molecules of a gas have high speeds associated with their Brownian motion, but these do not result in a bulk transfer of gas from one place to another. The flow velocity is thus defined as the average velocity of many

molecules. Similarly, the temperature is defined by the average energy of the Brownian motion. The density is defined by the mass of the average number of molecules to be found in a given volume. Other macroscopic properties, such as pressure and viscosity, likewise result from the average action of many molecules.

None of these averaging processes is meaningful unless the averaging is carried out over a large number of molecules. A fluid particle must thus be large enough to contain many molecules. It must still be effectively at a point with respect to the flow as a whole. Thus the continuum hypothesis can be valid only if there is a length scale, L_2, which we can think of as the size of a fluid particle, such that

$$L_1 \ll L_2 \ll L_3 \tag{5.1}$$

where the meanings of L_1 and L_3 are illustrated by Fig. 5.1. This figure uses the example of temperature, rather than the natural first choice of velocity, because it is easier to discuss a scalar. It shows schematically the average Brownian energy of the molecules in a volume L^3 plotted against the length scale L (on a logarithmic scale). The centre of the volume may be supposed fixed as its size is varied. When the volume is so small that it contains only a few molecules, there are large random fluctuations; the change produced by increasing the volume depends on the particular speeds of the new molecules then included. As the volume becomes large enough to contain many molecules, the fluctuations become negligibly small. A temperature can then meaningfully be defined. L_1 is proportional to, but an order of magnitude or so larger than, the average

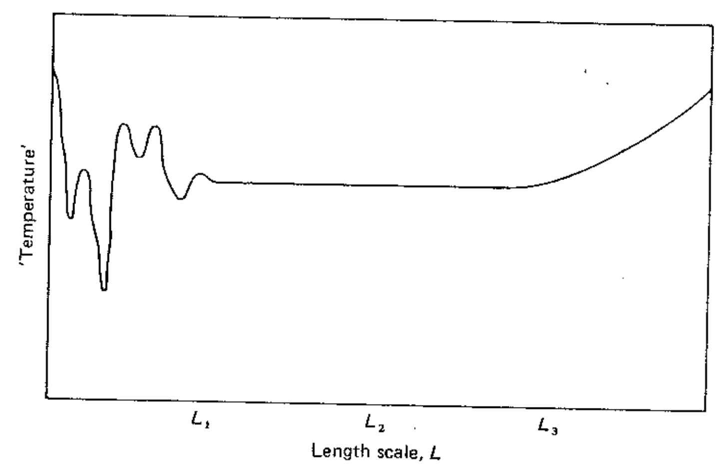

FIG. 5.1 Schematic variation of average energy of molecules with length scale. See text.

distance from a molecule to its nearest neighbour. At the other extreme, the volume may become so large that it extends into regions where the temperature is significantly different. This will result in an increase or decrease in the average. L_3 is a typical length scale associated with the flow; that is, a typical distance over which the macroscopic properties vary appreciably.

The applicability of the continuum hypothesis depends on there being a significant plateau between L_1 and L_3 as shown. One may regard L_2 as being an infinitesimal distance so far as macroscopic effects are concerned, and formulate the equations (as differential equations implicitly involving the limit of small separations) ignoring the behaviour on still smaller length scales.

The same fluid particle does not consist of just the same molecules at all times. The interchange of molecules between fluid particles is taken into account in the macroscopic equations by assigning to the fluid diffusive properties such as viscosity and thermal conductivity. For example (again considering a gas for simplicity) the physical process by which the velocity distribution of Fig. 1.1 generates the stress shown is the Brownian movement of molecules across AB; those crossing in, say, the $+y$-direction have on average less x-momentum and so tend to reduce the momentum of the fluid above AB. The same fluid particle may be identified at different times, once the continuum hypothesis is accepted, through the macroscopic formulation. This specifies (in principle) a trajectory for every particle and thus provides meaning to the statement that the fluid at one point at one time is the same as that at another point at another time. For example, for a fluid macroscopically at rest, it is obviously sensible to say that the same fluid particle is always in the same place—even though, because of the Brownian motion, the same molecules will not always be at that place.

However, for the continuum hypothesis to be plausible, it is evidently necessary for the molecules within a fluid particle to be strongly interacting with one another. If each molecule acted just as if the others were not there, there would be little point in identifying the aggregate as a particle. Thus, if λ is the molecular-mean-free path, continuum mechanics can be applied only if

$$\lambda \ll L_2 \tag{5.2}$$

so that each molecule undergoes many collisions whilst traversing a distance that can still be regarded as infinitesimal. Since λ can be large compared with L_1 as defined above, this is an additional requirement to (5.1).

Once the continuum hypothesis has been introduced, we can formulate the equations of motion on a continuum basis, and the molecular

structure of the fluid need not be mentioned any more. Hence, although the concepts developed above underlie the whole formulation, we shall not have much occasion to refer back to them. Velocity, henceforth, will be either a mathematical quantity or something (hopefully equivalent) that one measures experimentally. So will all the other parameters. Their definitions as averages over molecules provide answers to the implicit, but rarely explicit, questions: 'What is the real physical meaning of this mathematical quantity?' 'What quantity does one ideally wish to measure?'

The continuum hypothesis is only a hypothesis. The above discussion suggests that it is plausible, but nothing more. The real justification for it comes subsequently, through the experimental verification of predictions of the equations developed on the basis of the hypothesis.

5.3 Eulerian and Lagrangian coordinates

In setting up the equations governing the dynamics of a fluid particle, we evidently need to decide whether we should use coordinates fixed in space or coordinates that move with the particle. These two procedures are known respectively as the Eulerian and Lagrangian specifications. The equations are much more readily formulated using the former because the Lagrangian specification does not immediately indicate the instantaneous velocity field on which depend the stresses acting between fluid particles. Throughout this book we use only the Eulerian specification; i.e. we write the velocity

$$\mathbf{u} = \mathbf{u}(\mathbf{r}, t) \tag{5.3}$$

where $\mathbf{r}$ is the position in an inertial frame of reference and t is time. Values of $\mathbf{u}$ at the same $\mathbf{r}$ but different t do not, of course, correspond to the same fluid particle.

It is not always easy to relate Lagrangian aspects, such as the trajectories of fluid particles, to an Eulerian specification. In the context of this book, this is particularly relevant to the interpretation of flow visualization experiments in which dye marks certain fluid particles. The relationship of the observed patterns to the corresponding Eulerian velocity field may not be simple.

5.4 Continuity equation

We are now ready to start on the actual formulation of the equations. We consider first the representation of mass conservation, often called continuity.

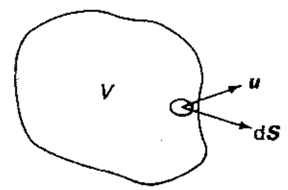

FIG. 5.2 Definition sketch for derivation of continuity equation.

Consider an arbitrary volume V fixed relative to Eulerian coordinates and entirely within the fluid (Fig. 5.2). Fluid moves into or out of this volume at points over its surface. If $d\boldsymbol{S}$ is an element of the surface (the magnitude of $d\boldsymbol{S}$ being the area of the element and its direction the outward normal) and $\boldsymbol{u}$ is the velocity at the position of this element, it is the component of $\boldsymbol{u}$ parallel to $d\boldsymbol{S}$ that transfers fluid out of V. Thus, the outward mass flux (mass flow per unit time) through the element is $\rho\boldsymbol{u}\cdot d\boldsymbol{S}$, where ρ is the fluid density. Hence,

$$\text{rate of loss of mass from } V = \int_S \rho\boldsymbol{u}\cdot d\boldsymbol{S}. \tag{5.4}$$

(This is, of course, negative if the mass in V is increasing.) We have also

$$\text{total mass in volume } V = \int_V \rho \, dV. \tag{5.5}$$

Hence

$$\frac{d}{dt}\int_V \rho \, dV = \int_V \frac{\partial\rho}{\partial t} dV = -\int_S \rho\boldsymbol{u}\cdot d\boldsymbol{S}. \tag{5.6}$$

We are interested in the mass balance at a point, rather than that over an arbitrary finite volume. Hence, we allow V to shrink to an infinitesimal volume; the integration in $\int(\partial\rho/\partial t)\,dV$ is redundant and we have

$$\frac{\partial\rho}{\partial t} = -\lim_{V\to 0}\left[\int \rho\boldsymbol{u}\cdot d\boldsymbol{S}/V\right]. \tag{5.7}$$

That is,

$$\frac{\partial\rho}{\partial t} = -\text{div}\,\rho\boldsymbol{u} \tag{5.8}$$

by definition of the operator div. This gives the general expression representing mass conservation for a fluid in which both $\boldsymbol{u}$ and ρ are functions of position,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0. \tag{5.9}$$

This is known as the continuity equation.

For the important special case when ρ is constant (see Section 1.2 (3) and Section 5.8) the continuity equation reduces to the very simple form

$$\nabla\cdot\boldsymbol{u} = 0. \tag{5.10}$$

The velocity field is solenoidal vector. We notice that this does not assume steady flow. The time-variation does not appear explicitly in the continuity equation of a constant density fluid even when the flow is unsteady.

5.5 The substantive derivative

The next equation to be derived is the representation of Newton's second law of motion; i.e. the rate of change of momentum of a fluid particle is equal to the net force acting on it. We need first of all an expression for the rate of change of momentum of a fluid particle. It would not be correct to equate the rate of change of momentum at a fixed point to the force, because different particles are there at different times. Even in steady flow, for example, a fluid particle can change its momentum by travelling to a place where the velocity is different; this acceleration requires a force to produce it.

This is one example of a general problem. On occasion, one needs to know the rate of change of other quantities whilst following a fluid particle. In problems where thermal effects are important, for example, various physical processes may heat or cool the fluid. These determine the rate of change of the temperature of a fluid particle, not the rate of change at a fixed point.

In this section, therefore, we examine the general question of rates of change following the fluid. It is easier in the first place to consider a scalar quantity, and we denote this by T, thinking of the example of temperature.

Quite generally, the small change δT produced by a small change δt in time and small changes δx, δy, δz in Cartesian position coordinates is

$$\delta T = \frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z \tag{5.11}$$

and a rate of change can be formulated by dividing by δt:

$$\frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial T}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial T}{\partial z}\frac{\delta z}{\delta t}. \tag{5.12}$$

If now we choose δx, δy, and δz to be the components of the small distance moved by a fluid particle in time δt, then (in the limit $\delta t \to 0$)

this is the rate of change of T of that particle. Also $\delta x/\delta t$, $\delta y/\delta t$ and $\delta z/\delta t$ are then (in the same limit) the three components of the velocity of the particle (u, v, and w). We thus have

$$\frac{\mathrm{D}T}{\mathrm{D}t}=\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}. \qquad (5.13)$$

In general $\mathrm{D}/\mathrm{D}t$ denotes the rate of change (of whatever quantity it operates on) following the fluid. This operator is known as the substantive derivative.

We can rewrite eqn (5.13) as

$$\frac{\mathrm{D}T}{\mathrm{D}t}=\frac{\partial T}{\partial t}+\boldsymbol{u}\cdot\nabla T \qquad (5.14)$$

and the operator in general as

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla \qquad (5.15)$$

(exhibiting the physically obvious fact that it does not depend on the particular coordinates used).

We see that the relationship combines the two ways in which the temperature of a fluid particle can change. It can change because the whole temperature field is changing—a process present even if the particle is at rest. And it can change by moving to a position where the temperature is different—a process present even if the temperature field as a whole is steady. As one would expect, this latter process depends on the magnitude of the spatial variations of the temperature and on the velocity, determining how quickly the fluid moves through the spatial variations.

Nothing in the above analysis restricts it to scalar quantities, and we can similarly write that the rate of change of a vector quantity $\boldsymbol{B}$ following a fluid particle is

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t}=\frac{\partial \boldsymbol{B}}{\partial t}+u\frac{\partial \boldsymbol{B}}{\partial x}+v\frac{\partial \boldsymbol{B}}{\partial y}+w\frac{\partial \boldsymbol{B}}{\partial z}=\frac{\partial \boldsymbol{B}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{B}. \qquad (5.16)$$

Whereas $\boldsymbol{u}\cdot\nabla T$ is the scalar product of vectors $\boldsymbol{u}$ and ∇T, $\boldsymbol{u}\cdot\nabla\boldsymbol{B}$ cannot be similarly split up. It is meaningful only as a whole. $(\boldsymbol{u}\cdot\nabla)$ operating on a vector must be thought of as a new operator (defined through its Cartesian expansion).

The particular case, $\boldsymbol{B}=\boldsymbol{u}$, gives the rate of change of velocity following a fluid particle,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}. \qquad (5.17)$$

The velocity $\boldsymbol{u}$ now enters in two ways, both as the quantity that changes as the fluid moves and as the quantity that governs how fast the change occurs. Mathematically, however, it is just the same quantity in both its roles.

Returning finally to the information that we require for the dynamical equation, we have that the rate of change of momentum per unit volume following the fluid is

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}. \qquad (5.18)$$

Why, the reader may ask, is it $\rho\,\mathrm{D}\boldsymbol{u}/\mathrm{D}t$ and not $\mathrm{D}(\rho\boldsymbol{u})/\mathrm{D}t$? (The distinction is important in the general case when both $\boldsymbol{u}$ and ρ are variables.) The only reason why a particular bit of fluid is changing its momentum is that it is changing its velocity. If it is simultaneously changing its density, this is not because it is gaining or losing mass, but because it is changing the volume it occupies. This change is therefore irrelevant to the momentum change. Expressing the distinction verbally instead of algebraically, we may say that '(the rate of change of momentum) per unit volume' is different from 'the rate of change of (momentum per unit volume)', and the former is the relevant quantity.

5.6 The Navier–Stokes equation

From above, the left-hand side of the dynamical equation, representing Newton's second law of motion, is $\rho\,\mathrm{D}\boldsymbol{u}/\mathrm{D}t$. The right-hand side is the sum of the forces (per unit volume) acting on the fluid particle. We now consider the nature of these forces in order to complete the equation.

Some forces are imposed on the fluid externally, and are part of the specification of the particular problem. One may need, for example, to specify the gravity field in which the flow is occurring. On the other hand, the forces due to the pressure and to viscous action, of which simple examples have been given in Section 2.2, are related to the velocity field. They are thus intrinsic parts of the equations of motion and have to be considered here. Both the pressure and viscous action generate stresses acting across any arbitrary surface within the fluid; the force on a fluid particle is the net effect of the stresses over its surface.

The generalization of the pressure force from the simple case considered in Section 2.2 is straightforward. We saw there that the net force per unit volume in the x-direction as a result of a pressure change in that direction is $-\partial p/\partial x$. For a general pressure field, similar effects act in all directions and the total force per unit volume is $-\mathrm{grad}\,p$. This term always appears in the dynamical equation; when a fluid is brought into motion, the pressure field is changed from that existing when it is at rest (the hydrostatic pressure). We can regard this for the moment as an experimental result. We shall be seeing that it is necessary to have the pressure as a variable in order that the number of variables matches the number of equations.

The general form of the viscous force is not so readily inferred from any simple example. The mathematical formulation is outlined as an appendix to this chapter. Here we shall look at some of the physical concepts underlying viscous action, and then quote the expression for the viscous term in the dynamical equation that is given by rigorous formulation of these concepts. (Some further discussion of the physical action of viscosity will be given in the context of the particular example of jet flow in Section 11.7.)

Viscous stresses oppose relative movements between neighbouring fluid particles. Equivalently, they oppose the deformation of fluid particles. The difference between these statements lies only in the way of verbalizing the rigorous mathematical concepts, as is illustrated by Fig. 5.3. The change in shape of the initially rectangular region is produced by the ends of one diagonal moving apart and the ends of the other moving together. As the whole configuration is shrunk to an infinitesimal one, it may be said either that the particle shown is deforming or that particles on either side of AB are in relative motion. The rate of deformation depends on the velocity gradients in the fluid. The consequence of this behaviour is the generation of a stress (equal and opposite forces on the two sides) across a surface such as AB; this stress depends on the properties of the fluid as well as on the rate of deformation.

The stress can have any orientation relative to the surface across which it acts. The special case, considered in Sections 1.2(4) and 2.2, in which the stress is in the plane of the surface is often thought of as the 'standard' case. We therefore look for a moment at a simple situation in which viscous stresses normal to the surface govern the behaviour. This is the falling column produced for example when a viscous liquid is poured from a container (Fig. 5.4). In the absence of side-walls transverse viscous stresses cannot be generated as they are in channel flow. The

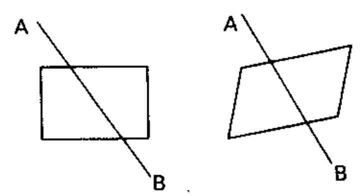

FIG. 5.3 Deformation of rectangular element and relative motion of fluid on either side of arbitrary line through the element.

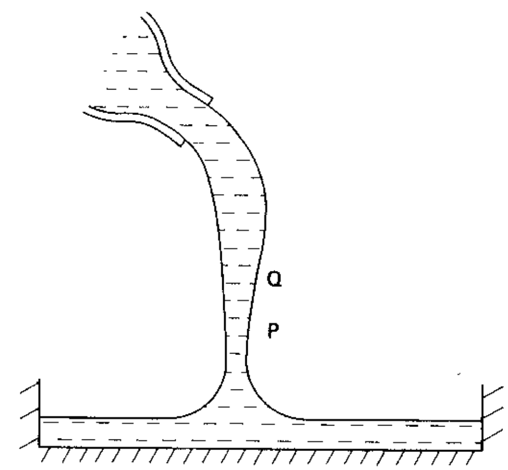

FIG. 5.4 Pouring of viscous liquid.

reason the fluid at, say, P does not fall with an acceleration of g is the viscous interaction with the more slowly falling fluid at Q.

In the general case, the stress is a quantity with a magnitude and two directions, the direction in which it acts and the normal to the surface, associated with it. It is thus a second-order tensor. (The stresses acting across surfaces of different orientations through the same point are not, or course, independent of one another.) The rate of deformation is also expressed by a second-order tensor—the rate-of-strain tensor. From the considerations above we expect this to involve the velocity gradients. However, not all distributions of velocity variation lead to deformation; a counter-example is the rotation of a body of fluid as if it were rigid (see Section 6.4). The rate-of-strain tensor selects the appropriate features of the velocity field.

One expects the stress tensor to depend on the rate-of-strain tensor and on the properties of the fluid. A Newtonian fluid (see Section 1.2(4)) can now be defined rigorously as one in which the stress tensor and the rate-of-strain tensor are linearly related.

The remaining ideas contained in the derivation of the viscous term of the dynamical equation are simply symmetry considerations. For example, a mirror-image flow pattern must generate a mirror-image stress distribution. And the analysis of a flow configuration using different coordinates must give the same result.

In Cartesian coordinates, the x-component of the viscous force per unit volume (see appendix to this chapter) is

$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\nabla\cdot\boldsymbol{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]. \quad (5.19)$$

Similar expressions for the y- and z-components are given by appropriate permutations.

Here μ is the coefficient of viscosity, defined through the special case considered in Section 1.2(4). λ is a second viscosity coefficient. One would expect there to be a second such coefficient, independent of the first, by analogy with the fact that there are two independent elastic moduli. This is a valid analogy. However, it has often been the practice to introduce a relationship between μ and λ ($\lambda = -2\mu/3$). This is done at the cost of redefining the pressure so that it is not the thermodynamic pressure, and the second independent parameter then appears in the relationship between the two pressures [26,324]. λ is difficult to measure experimentally and is not known for the variety of fluids for which there are values of μ. Hence, the statement that a fluid is Newtonian usually means that μ is observed to be independent of the rate of strain and that λ is assumed to be so too.

However, for a fluid of constant density, the continuity equation (5.10) causes the term involving λ to drop out. If, additionally, μ is taken to be a constant, expression (5.19) reduces (with a further use of (5.10)) to simply

$$\mu\nabla^2 u. \tag{5.20}$$

The y- and z-components are correspondingly $\mu\nabla^2 v$ and $\mu\nabla^2 w$, and the vectorial viscous force per unit volume is $\mu\nabla^2\boldsymbol{u}$.†

Because (5.20) is so much simpler than (5.19), one prefers to use the former whenever possible; one tends to make this approximation even when the density and/or the viscosity do vary appreciably (see also appendix to Chapter 14). That will be the procedure throughout this book. But it should be remembered that situations may arise in the laboratory or in a practical application which are properly described only by the full expression.

Collecting together the various contributions mentioned, we have

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \mu\nabla^2\boldsymbol{u} + \boldsymbol{F}. \tag{5.21}$$

This is known as the Navier–Stokes equation. It is the basic dynamical equation expressing Newton's second law of motion for a fluid of constant density.

The term $\boldsymbol{F}$ represents the contribution of those forces (such as gravity) mentioned at the beginning of this section that have to be included in the specification of the problem. This is often known as the body force term, because such forces act on the volume of a fluid

† The meaning of ∇^2 operating on a vector is defined through its Cartesian expansion.

particle, not over its surface in the way the stresses between fluid particles act. The reaction to a body force is remote from the fluid particle concerned: usually outside the fluid region, although occasionally on distant fluid particles.

We shall often be considering problems in which $\boldsymbol{F} = 0$, the cause of motion being either imposed pressure differences or relative movement of boundaries. No body forces are applied. It might be objected that, although one can well imagine all other sources of body force being eliminated, almost every flow will take place in a gravity field. It can be shown that gravitational body forces act significantly only on density differences. If the density is uniform, the gravitational force is balanced by a vertical pressure gradient which is present whether or not the fluid is moving and which does not interact with any flow. This hydrostatic balance can be subtracted out of the dynamical equation and the problem reduced to one without body forces. This assumes, of course, that the fluid region is supported at the bottom; flow under gravity down a vertical pipe provides an obvious example where this is not so and where the above remarks do not apply. We will consider the justification for subtracting out the hydrostatic balance more formally in Section 14.2.

The Navier–Stokes equation may be rewritten

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u} + \frac{1}{\rho}\boldsymbol{F}, \tag{5.22}$$

where $\nu = \mu/\rho$ and is a property of the fluid called the kinematic viscosity.

We notice that the equation is a non-linear partial differential equation in $\boldsymbol{u}$. The non-linearity arises from the dual role of the velocity in determining the acceleration of a fluid particle, as mentioned in Section 5.5. This non-linearity is responsible for much of the mathematical difficulty of fluid dynamics, and is the principal reason why our knowledge of the behaviour of fluids in motion is obtained in many cases from observation (both of laboratory experiments and of natural phenomena) rather than from theoretical prediction. The physical counterpart of the mathematical difficulty is the variety and complexity of fluid dynamical phenomena; without the non-linearity the range of these would be much more limited.

The continuity eqn (5.10) and the Navier–Stokes eqn (5.21) constitute a pair of simultaneous partial differential equations. Both represent physical laws which will always apply to every fluid particle. Together they provide one scalar equation and one vector equation—effectively four simultaneous equations—for one scalar variable (the pressure) and one vector variable (the velocity)—effectively four unknown quantities. The number of unknowns is thus correctly matched to the number of equations. We see that the pressure must necessarily be an intrinsic variable in fluid dynamical problems for there to be enough variables to satisfy the basic laws of mechanics.

For many particular problems it is convenient to use the equations referred to a coordinate system rather than the vectorial forms. Most often one uses Cartesian coordinates, but sometimes a curvilinear system is suggested by the geometry. Listed below are the forms taken by the continuity eqn (5.10) and the three components of the Navier–Stokes eqn (5.21) in Cartesian, cylindrical polar, and spherical polar co-ordinates:

1. Cartesian coordinates:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{5.23}$$

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]+F_x \tag{5.24}$$

together with similar equations for v and w.

2. Cylindrical polar coordinates (r = distance from axis, ϕ = azimuthal angle about axis, z = distance along axis):

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\phi}{\partial \phi}+\frac{\partial u_z}{\partial z}=0 \tag{5.25}$$

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\phi^2}{r}\right]=-\frac{\partial p}{\partial r}$$
$$+\mu\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2}+\frac{\partial^2 u_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right]+F_r \tag{5.26}$$

$$\rho\left[\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_r u_\phi}{r}+\frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi}+u_z\frac{\partial u_\phi}{\partial z}\right]=-\frac{1}{r}\frac{\partial p}{\partial \phi}$$
$$+\mu\left[\frac{\partial^2 u_\phi}{\partial r^2}+\frac{1}{r}\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{\partial^2 u_\phi}{\partial z^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right]+F_\phi \tag{5.27}$$

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi}+u_z\frac{\partial u_z}{\partial z}\right]=-\frac{\partial p}{\partial z}$$
$$+\mu\left[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2}+\frac{\partial^2 u_z}{\partial z^2}\right]+F_z. \tag{5.28}$$

3. Spherical polar coordinates (r = distance from origin, θ = angular displacement from reference direction, ϕ = azimuthal angle about line

$\theta = 0$):

$$\frac{\partial u_r}{\partial r}+\frac{2u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\theta \cot\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}=0 \tag{5.29}$$

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\theta^2}{r}-\frac{u_\phi^2}{r}\right]$$
$$=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial u_r}{\partial r}-\frac{2u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_r}{\partial \theta}\right.$$
$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial \phi^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}-\frac{2u_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right]+F_r \tag{5.30}$$

$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_r u_\theta}{r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}-\frac{u_\phi^2\cot\theta}{r}\right]$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial^2 u_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2\sin^2\theta}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}\right.$$
$$\left.+\frac{\cot\theta}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial \phi^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right]+F_\theta \tag{5.31}$$

$$\rho\left[\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_r u_\phi}{r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\theta u_\phi\cot\theta}{r}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right]$$
$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+\mu\left[\frac{\partial^2 u_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial u_\phi}{\partial r}-\frac{u_\phi}{r^2\sin^2\theta}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_\phi}{\partial \theta}\right.$$
$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right]+F_\phi. \tag{5.32}$$

(One sometimes requires also the individual stress components in polar coordinates. These will be introduced as required. For a more systematic treatment the reader is referred to Refs. [11] and [22].)

5.7 Boundary conditions

Since the governing equations of fluid motion are differential equations, the specification of any problem must include the boundary conditions. We would expect this on physical grounds; the motion throughout a fluid region is evidently influenced by the presence and motion of walls or other boundaries. We examine now the form taken by the conditions on the velocity field applying at boundaries.

There are obviously various types of boundary, giving rise to different possible conditions. However, the only case that we need consider in any detail for the purposes of this book is the most common type of boundary to a fluid region—the rigid impermeable wall.

One condition applying at such a wall is obviously provided by the requirement that no fluid should pass through the wall. If the wall is moving with velocity $\boldsymbol{U}$ and the velocity of a fluid particle right next to the wall is $\boldsymbol{u}$, then this means that the normal components of these two velocities must be the same:

$$\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = \boldsymbol{U} \cdot \hat{\boldsymbol{n}}, \tag{5.33}$$

where $\hat{\boldsymbol{n}}$ is the unit normal to the surface (Fig. 5.5). One often chooses a frame of reference in which the boundaries are at rest, giving $\boldsymbol{U} = 0$; this boundary condition then becomes

$$\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = 0 \tag{5.34}$$

or in Cartesian coordinates with x and z in the local tangential plane to the wall and y normal to it,

$$v = 0. \tag{5.35}$$

Another condition is provided by the no-slip condition, already mentioned in Section 2.2, that there should be no relative tangential velocity between a rigid wall and the fluid immediately next to it. Formally,

$$\boldsymbol{u} \times \hat{\boldsymbol{n}} = \boldsymbol{U} \times \hat{\boldsymbol{n}} \tag{5.36}$$

and, when $\boldsymbol{U} = 0$,

$$\boldsymbol{u} \times \hat{\boldsymbol{n}} = 0, \tag{5.37}$$

or

$$u = w = 0. \tag{5.38}$$

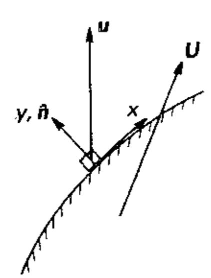

FIG. 5.5 Velocity vectors of solid and of fluid particle immediately next to its surface. (Note: $\boldsymbol{U}$ and $\boldsymbol{u}$ are shown different for definition purposes, although the text subsequently shows them to be the same.)

It is apparent that some such condition must pertain. For example, without it, there would be no boundary condition on u (eqns (2.7) and (2.16)) for pipe or channel flow and the solutions (eqns (2.8) and (2.17)) could not be obtained; viscous action would place no limit on the amount of fluid per unit time that could pass through a pipe under a given pressure gradient. However, it is not so apparent that the condition should take this exact form. The notion underlying the no-slip condition is that the interaction between a fluid particle and a wall is similar to that between neighbouring fluid particles. Within a fluid there cannot be any finite discontinuity of velocity. This would involve an infinite velocity gradient and so produce an infinite viscous stress that would destroy the discontinuity in an infinitesimal time. If, therefore, the wall acts like further fluid, the action of viscosity prevents a discontinuity in velocity between the wall and fluid; the no-slip condition must apply. However, this concept that the wall acts like further fluid is itself an assumption and the justification for the no-slip condition lies ultimately in experimental observation. This experimental justification takes two forms. The first is direct observation; dye or smoke introduced very close to a wall does stay at rest relative to the wall. The second is an *a posteriori* justification; the no-slip condition is assumed and the solutions of the equations found in simple cases; agreement between theory and experiment then justifies the original assumption. The no-slip condition is found to be violated only when the molecular-mean-free path becomes comparable with the distances involved; then the continuum equations are ceasing to be applicable anyway.

Conditions (5.33) and (5.36) in combination do, of course, give

$$\boldsymbol{u} = \boldsymbol{U} \tag{5.39}$$

or, in Cartesian coordinates on a wall at rest,

$$u = v = w = 0. \tag{5.40}$$

The total boundary condition is simply that there is no relative motion between a wall and the fluid next to it. It is, however, important to note (and it will be of significance subsequently, in Section 8.3) that the physical origins of the two parts of the condition are quite different. The no-slip condition depends essentially on the action of viscosity, whilst the impermeability condition does not.

This is a convenient point to mention the forces exerted by a moving fluid on a rigid boundary, a matter of obvious practical importance. This again uses the notion that the fluid acts on a wall in the same way as it acts on further fluid. However, no assumption is involved here; the stresses must be continuous or a fluid particle at a wall would experience infinite acceleration. Thus we may use the expressions in Section 5.6 for stresses in the interior of the fluid. We use Cartesian coordinates as above with the wall at $y=0$. Then the viscous stresses in the x-, y- and z-directions can be extracted from expression (5.19) and are given in the first place as

$$\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)_{y=0}; \quad 2\mu\left(\frac{\partial v}{\partial y}\right)_{y=0}; \quad \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)_{y=0}. \tag{5.41}$$

Since, from the continuity equation,

$$\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}-\frac{\partial w}{\partial z} \tag{5.42}$$

and since, also,

$$u=v=w=0 \quad \text{at} \quad y=0 \quad \text{for all } x \text{ and } z \tag{5.43}$$

these reduce to

$$\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}; \quad 0; \quad \mu\left(\frac{\partial w}{\partial y}\right)_{y=0}. \tag{5.44}$$

The first and third quantities are the tangential forces per unit area on the wall. There is no normal viscous force, but there is a pressure force of p per unit area in the $-y$-direction.

There are other important types of boundary besides the rigid impermeable wall. The free surface of a liquid is an obvious example. And a rigid wall with suction or injection of fluid through it has important practical applications, such as some aircraft wings where the flow is controlled by suction of the air. However, we do not need to formulate the corresponding boundary conditions for the purposes of this book.

One other case does need mention—when the boundary condition is applied at infinity. Often any boundaries are far from the region of interest, as in the examples of an aeroplane well above the ground or a small model placed in a wind-tunnel. The motion far from such an obstacle is the same as in the absence of the obstacle, and one has a boundary condition of the form

$$u \to u_0 \quad \text{as} \quad r \to \infty. \tag{5.45}$$

5.8 Condition for incompressibility

We have seen that both the continuity equation and the dynamical equation simplify greatly if one can treat the density, ρ, as a constant. Fortunately, there are many situations in which it is a good approximation to do so. Brief consideration has been given to this matter in Section

1.2(3) and we now examine more fully the conditions in which one may make this approximation.

The status of the equation

$$\rho = \text{const.} \tag{5.46}$$

is that of an equation of state. That is to say, in circumstances where it is not a good approximation, one needs instead an equation of state giving the density as a function of pressure and temperature.

Correspondingly, when one does take the density as being constant, one is saying that the density variations produced by the pressure and temperature variations are sufficiently small to be unimportant. In this section we shall be principally concerned with the effect of pressure variations, as we have already seen that such variations are intrinsic to any flow. We shall derive a criterion for these to have a negligible effect. Non-fulfilment of this criterion is the most familiar reason for departures from eqn (5.46); consequently this equation is called the incompressibility condition, although the name is not a complete summary of the requirements for it to be applicable.

Temperature variations are also in principle intrinsic to any flow. (They may also be introduced specifically but we leave to Section 14.2 and the appendix to Chapter 14 the corresponding considerations for that case.) Firstly, the expansions and contractions as the fluid moves through the pressure field involve temperature changes. The effect of these will, however, be covered by the following discussion of the pressure effect; nowhere will it be assumed that the changes are isothermal. Secondly, viscous action involves the dissipation of mechanical energy (see Section 11.7), which reappears as heat; we will consider this briefly at the end of this section.

Liquids are known to change their density very little even for large pressure changes. One would expect to be able to treat these as incompressible. It is less apparent that there are important circumstances in which gases can be so treated, although the result is perhaps not wholly unexpected if one recalls that the fractional change in the atmospheric pressure (and so the fractional change in the air density) is small even when strong winds are blowing. The following derivation of the criterion is thus of importance primarily for gases, although it is in fact quite general.

We can write the density

$$\rho = \rho_0 + \Delta\rho \tag{5.47}$$

where ρ_0 is a reference density—for example, the density at some arbitrarily chosen point—and $\Delta\rho$ is the local departure from this. If

$$\Delta\rho/\rho_0 \ll 1 \tag{5.48}$$

then, for example, the term $\rho\,\mathrm{D}\boldsymbol{u}/\mathrm{D}t$ in the dynamical equation can be approximated by $\rho_0\,\mathrm{D}\boldsymbol{u}/\mathrm{D}t$. A similar comment applies to the other places where ρ appears in the continuity and dynamical equations. Thus the equations with ρ const. may be used when relationship (5.48) is satisfied.

Since the density changes under consideration result from pressure variations, in order to estimate the typical size of $\Delta\rho$ we need to know the typical size ΔP of these pressure variations. We get this information from the requirement that the pressure force must be balanced by other terms in the Navier–Stokes equation and thus will be of the same order of magnitude as at least one other term. (If the flow is produced by imposed pressure differences, then at the start of the motion the fluid will accelerate until terms involving the velocity become comparable with the pressure force. If the flow is produced by imposed velocity differences, then the flow will generate pressure differences of an appropriate size.) We confine attention to steady flow without body forces, and so, either

$$\nabla p \sim \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} \qquad (5.49)$$

or

$$\nabla p \sim \mu\nabla^2\boldsymbol{u} \qquad (5.50)$$

or both (with the symbol ~ meaning 'is of the same order of magnitude as'). We shall pursue the consequences of (5.49). We shall see in Chapter 8 that the only circumstances when (5.50) applies whilst (5.49) does not are when the Reynolds number is low, and the following analysis does not then apply.

Provided that the x-axis is chosen in a direction in which significant variations occur, (5.49) can be written

$$\frac{\partial p}{\partial x} \sim \rho u\frac{\partial u}{\partial x} = \tfrac{1}{2}\rho\frac{\partial u^2}{\partial x}. \qquad (5.51)$$

This indicates that

$$\Delta P/L \sim \rho\Delta(U^2)/L \qquad (5.52)$$

where ΔP and $\Delta(U^2)$ are typical differences in p and u^2 between points a distance L apart. This means that, if one arbitrarily chose many such pairs of points, the average difference in p would be of the general size ΔP. Since ΔP and $\Delta(U^2)$ are defined only as order-of-magnitude quantities they do not require more precise definition than that.

If L is the general length scale of the flow, ΔP and $\Delta(U^2)$ are the orders of magnitude of the variations of p and u^2 within the fluid; and are related by

$$\Delta P \sim \rho\Delta(U^2). \qquad (5.53)$$

We do not need to maintain the distinction between a typical difference in the (square of the) velocity and a typical value of the (square of the) velocity itself; i.e. we can write

$$\Delta U \sim U; \quad \Delta(U^2) \sim U^2 \qquad (5.54)$$

where U is a velocity scale. The reason for this is that one always can (and normally will) choose a frame of reference in which some points of the flow, for example those at a boundary, are at rest. (It would be perverse to analyse the dynamics of a low-speed aeroplane from the frame of reference of a high-speed aeroplane.) On the other hand, it is necessary to maintain the distinction between the pressure difference scale and the pressure itself. Since the pressure appears only in the form ∇p in the governing equations, the absolute pressure can be increased indefinitely without directly† altering the dynamics; only pressure differences are relevant.

Thus we have

$$\Delta P \sim \rho U^2 \qquad (5.55)$$

which indicates the typical pressure variation in a flow of typical speed U. We now use this to determine the typical density variation. This depends on the fluid and in particular on its compressibility, β;

$$\Delta\rho/\rho \sim \beta\Delta P. \qquad (5.56)$$

(For order of magnitude considerations it does not matter whether β is the isothermal compressibility, the adiabatic compressibility or what.) The final result is given in a convenient form if we now introduce the speed of sound, a, in the fluid:

$$a^2 = \left(\frac{\partial p}{\partial\rho}\right)_S \sim \frac{1}{\rho\beta} \qquad (5.57)$$

(S = entropy). Here a is introduced simply as a property of the fluid under consideration, a measure of its compressibility.

Combining the various relationships, we have

$$\frac{\Delta\rho}{\rho} \sim \frac{\Delta P}{\rho a^2} \sim \frac{U^2}{a^2}. \qquad (5.58)$$

Thus criterion (5.48) is fulfilled if

$$(\mathrm{Ma})^2 = U^2/a^2 \ll 1. \qquad (5.59)$$

Flows at speeds low compared with the speed of sound in the fluid thus behave as if the fluid were incompressible.

† It can do so indirectly by changing fluid properties.

The ratio U/a is known as the Mach number of the flow, and incompressible flows thus occur at low Mach number. The fact that $(\mathrm{Ma})^2$ is involved in relationship (5.59) means that Ma does not have to be very small; when Ma is less than about 0.2, density variations are only a few per cent, bringing the accuracy of the incompressibility assumption to within the sort of accuracy attainable in many fluid dynamical investigations.

Many important gas flows do occur at low Mach number. For example, the speed of sound in air under atmospheric conditions is around $300\,\mathrm{m\,s^{-1}}$. Evidently, one will often be concerned with speeds low compared with this.

The fact that liquids are much less compressible than gases is contained in this analysis by the fact that they have much higher sound speeds, thus giving lower Mach numbers at the same U.

One general comment about the nature of the above argument may be made. The way in which ρ enters the governing equations is important in deciding that relationship (5.48) is a justification for treating ρ as a constant. Pressure provides an immediate counter-example in the present context. The fractional pressure change may also be small; in fact, for a gas,

$$\Delta P/p \sim \Delta\rho/\rho. \tag{5.60}$$

This does not mean that the pressure can be treated as a constant. As we have already noted, the pressure appears only in ∇p and the absolute pressure can be altered at will without changing the equations; comparison with it is thus irrelevant.

It was remarked that the above analysis does not apply at low Reynolds number. Then one has to use (5.50) instead of (5.49). The corresponding analysis then gives the criterion for incompressibility as

$$(\mathrm{Ma})^2 \ll \mathrm{Re} \tag{5.61}$$

where Re is the Reynolds number—a somewhat academic result, rarely relevant to real situations. (But see Section 26.5.)

A similar treatment of viscous dissipation shows that it also frequently has negligible effect. (It is, for example, not noticeably warmer swimming at the bottom of a waterfall than at the top.) The details will not be given here; the corresponding matter in the topic of free convection is discussed in the appendix to Chapter 14. As in that case, the criterion for the resulting density changes to be negligible can be expressed in a form not involving the viscosity coefficient and related to the criterion for incompressibility. This effect can be ignored wherever (5.59) is fulfilled. Hence, this provides an adequate criterion for the use of the incompressible flow equations.

In general, the character of low Mach number flow may be summarized by saying that, from a thermodynamic point of view, the whole flow is only a perturbation. This remark is most directly applicable to gases—the corresponding considerations for liquids are more complicated—and in fact we use the properties of a perfect gas to illustrate it. Then (by putting the appropriate equation of state into (5.57))

$$a^2 = \gamma RT \tag{5.62}$$

where γ is the ratio of specific heat capacities, R the gas constant (the universal gas constant divided by the molecular weight, see comment in the Notation section, p. 468) and T the temperature. Also the internal energy per unit mass (often denoted by U in texts in thermodynamics) is

$$E = C_V T. \tag{5.63}$$

γR and C_V are of the same order of magnitude. Hence, when $U^2 \ll a^2$, the flow kinetic energy per unit volume is small compared with the internal energy per unit volume

$$\tfrac{1}{2}\rho U^2 \ll \rho E. \tag{5.64}$$

Thus, for example, if all the energy of a flow were dissipated viscously, the net change in internal energy would be fractionally small. More generally none of the processes involved in low Mach number flow involve major changes in thermodynamic state. This is just what is required for a gas to have only small fractional changes in density.

Incidentally, it follows from the above that the flow velocity in low Mach number flow of a gas is small compared with the typical Brownian velocity of a molecule. Any individual molecule is moving much faster as a result of its thermal motions than as a result of the flow. It is only because the Brownian velocity averages very closely indeed to zero when one considers a large number of molecules that the flow emerges as a significant net effect.

Thus there is a sense in which all incompressible flows are 'only a perturbation'. But that remark will scarcely impress someone standing in a wind of 100 km/hour ($\mathrm{Ma} \approx 0.1$)!

Appendix: Derivation of viscous term of dynamical equation

This appendix presents the main points of the mathematical formulation of the ideas described physically in Section 5.6. For a more complete treatment, see, for example, Refs. [11, 14, 19, 20, 26]. We consider briefly each of the stress tensor and rate of strain tensor, and then

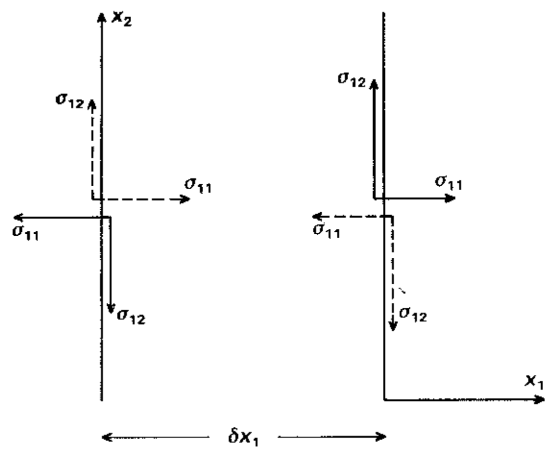

FIG. 5.6 The stress tensor: see text.

determine the consequences of a linear relationship between them. The summation convention for repeated algebraic suffixes applies throughout.

Two of the nine components of the stress tensor σ_{ij} in Cartesian coordinates (x_1, x_2, x_3) are shown in Fig. 5.6. We consider for the moment just the left-hand half of this figure, where the stresses acting across a surface normal to the x_1 direction are shown (σ_{13} is normal to the page). The same convention as in Fig. 1.1 is adopted, that the side of the surface of the fluid on which the stress acts is indicated by a slight displacement of the roots of the arrows; stresses on opposite sides are, of course, exactly equal and opposite. The first suffix on each component indicates the orientation of the surface across which it acts and the second indicates the direction in which it acts (with the sign convention that σ_{ij} is positive when the stress on the lower x_i side is in the positive x_j direction). Some authors use the suffixes the other way round; since the stress tensor can be shown to be symmetric (otherwise infinite angular accelerations would arise) this does not much matter. Definition of nine components in this way provides a complete specification of the stress; components acting across surfaces of other orientations are given by the usual rules for the effect of coordinate rotation.

The total stress acting in a fluid has contributions from both pressure and viscous effects,

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}. \tag{5.65}$$

Inclusion of the right-hand half of Fig. 5.6 extends it analogously to the extension from Fig. 1.1 to Fig. 2.3. The net force on a fluid particle is

given by the differences in the stresses acting across opposite faces. The solid arrows in Fig. 5.6 are those acting on the fluid between the two surfaces. For example, a net force in the x_1-direction arises from the difference in the two values of σ_{11}. However, forces in this direction arise also from variations of σ_{21} in the x_2-direction and of σ_{31} in the x_3-direction. Consequently, the total force per unit volume in the x_j-direction is

$$\frac{\partial \sigma_{ij}}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_i}. \tag{5.66}$$

What determines the viscous stress τ_{ij}? Figure 5.7 illustrates the effect of velocity gradients in the fluid. Two material points A and B are instantaneously at vectorial positions x_i and $x_i + \delta x_i$ and thus separated by δx_i. They are moving with velocities u_i and $u_i + \delta u_i$ so that after a time δt they are at A′ and B′ separated by $\delta x_i + \delta u_i\, \delta t$. The strain is related to the change $\delta u_i\, \delta t$ normalized by the original separation δx_i. (Doubling δx_i will double δu_i provided that both are infinitesimal.) Hence, rates of strain are related to the velocity gradient tensor

$$\zeta_{ij} = \partial u_i / \partial x_j. \tag{5.67}$$

However, there are some velocity gradient fields that involve no changes in the length of any material line, and thus no distortions and no viscous effects. Suppose the length of AB is δl,

$$(\delta l)^2 = \delta x_i\, \delta x_i \tag{5.68}$$

and so

$$\frac{\mathrm{D}(\delta l)^2}{\mathrm{D}t} = 2\,\delta x_i \frac{\mathrm{D}(\delta x_i)}{\mathrm{D}t} = 2\delta x_i\, \delta u_i$$

$$= 2\,\delta x_i\, \delta x_j \frac{\partial u_i}{\partial x_j} = \delta x_i\, \delta x_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \delta x_i\, \delta x_j (\zeta_{ij} + \zeta_{ji}). \tag{5.69}$$

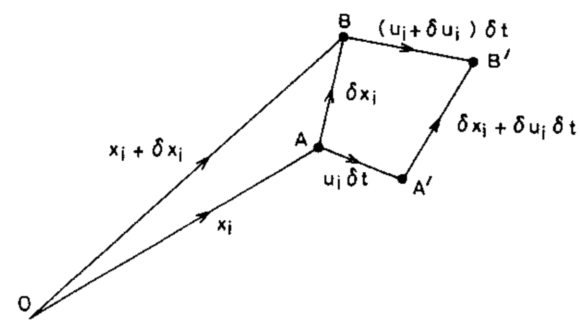

FIG. 5.7 Relative movements in flow: see text.

It is thus the symmetrical combinations of the velocity gradients that give rise to rates of strain. (A motion in which

$$\frac{\partial u_i}{\partial x_j} = -\frac{\partial u_j}{\partial x_i} \tag{5.70}$$

for all i and j does not involve any distortions. It is in fact some combination of uniform translation and rigid body rotation. The step from the fourth to the fifth expression in (5.69) makes explicit the otherwise not immediately apparent fact that the summation over i and j in the fourth expression involves terms that cancel when (5.70) is true.) Hence, the rate of strain tensor e_{ij} is the symmetric part of ζ_{ij},

$$e_{ij} = \tfrac{1}{2}(\zeta_{ij} + \zeta_{ji}). \tag{5.71}$$

(Parenthetically, the antisymmetric part

$$\eta_{ij} = \tfrac{1}{2}(\zeta_{ij} - \zeta_{ji}) \tag{5.72}$$

corresponds to the vorticity—see Section 6.4;

$$\eta_{ij} = -\tfrac{1}{2}\,\epsilon_{ijk}\omega_k.) \tag{5.73}$$

For a Newtonian fluid τ_{ij} is linearly related to e_{ij}:

$$\tau_{ij} = \Lambda_{ijkl}e_{kl}. \tag{5.74}$$

The physical processes must be independent of the orientation and handedness of the axes. Λ_{ijkl} must thus be an isotropic tensor and the most general form it can take is [212]

$$\Lambda_{ijkl} = \lambda\,\delta_{ij}\,\delta_{kl} + \xi\,\delta_{ik}\,\delta_{jl} + \chi\,\delta_{il}\,\delta_{jk}. \tag{5.75}$$

This gives

$$\tau_{ij} = \lambda\,\delta_{ij}e_{kk} + (\xi + \chi)e_{ij} \tag{5.76}$$

(since $e_{ij} = e_{ji}$). There are thus two arbitrary constants involved; these are physical properties of the particular fluid. From the particular case $e_{12} = \frac{1}{2}\,\partial u/\partial y$ and all other e_{ij} equal to zero, we can identify that

$$\xi + \chi = 2\mu \tag{5.77}$$

where μ is the coefficient of viscosity introduced in Section 1.2. Also,

$$e_{kk} = \frac{\partial u_k}{\partial x_k} = \operatorname{div} \boldsymbol{u} \tag{5.78}$$

and so

$$\tau_{ik} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\delta_{ij}\operatorname{div}\boldsymbol{u}. \tag{5.79}$$

From (5.66) the viscous force per unit volume is $\partial\tau_{ij}/\partial x_i$. Putting (5.79) into this and writing it in expanded form gives expression (5.19).

6

FURTHER BASIC IDEAS

6.1 Streamlines, streamtubes, particle paths and streaklines [388]

A streamline is defined as a continuous line within the fluid of which the tangent at any point is in the direction of the velocity at that point. Its relationship to the velocity field is thus analogous to the relationship of a line of force to an electric field. Patterns of streamlines are useful (particularly in two-dimensional flow) in providing a pictorial representation of a flow.

The streamlines for a known velocity field (u, v, w) are given as solutions of the pair of differential equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{6.1}$$

Two streamlines cannot intersect except at a position of zero velocity; otherwise one would have the meaningless situation of a velocity with two directions.

A streamtube is a tubular region within the fluid bounded by streamlines. Because streamlines cannot intersect, the same streamlines pass through a streamtube at all stations along its length.

Consider two stations along a streamtube of cross-sectional areas S_1 and S_2 as in Fig. 6.1. We suppose that the cross-sections are small enough that there is negligible variation of physical quantities over them and we can say that the densities and speeds at the two stations are ρ_1 and q_1 and ρ_2 and q_2 ($q = |\boldsymbol{u}|$; we can use the scalar quantity, as the direction is by definition along the streamtube). The rate at which mass is entering the volume between the two stations is $\rho_1 q_1 S_1$; the rate at which it is leaving is $\rho_2 q_2 S_2$. If the flow is either steady or incompressible (or, but not necessarily, both) the mass in this region is not changing, and so

$$\rho_1 q_1 S_1 = \rho_2 q_2 S_2. \tag{6.2}$$

If the flow is incompressible, $\rho_1 = \rho_2$ and

$$q_2/q_1 = S_1/S_2 \tag{6.3}$$

or, for a general station along the streamtube,

$$q \propto 1/S. \tag{6.4}$$

The speed is inversely proportional to the cross-sectional area. Hence requires equality between systems of both the Reynolds number and the Mach number, $\mathrm{Ma} = U/a$ (a is the speed of sound; see Section 5.8). Thus

$$C_D = f(\mathrm{Re}, \mathrm{Ma}). \tag{7.26}$$

4. When the flow is unsteady as a result of changes in the imposed conditions, these changes will have a time scale Ψ associated with them. In problems such as the above there is then the additional non-dimensional parameter $U\Psi/L$, and dynamical similarity throughout the development of the flow requires equality of this in addition to the Reynolds number.

It should be noted that, in the context of model testing, the above discussion of dynamical similarity is the statement of an ideal. It is often not possible in practice to make all the governing non-dimensional parameters the same as on the full scale. In ship model testing, for instance, a reduction in L requires an increase in U to keep the Reynolds number the same but a reduction in U to keep the Froude number the same (since there is little manoeuvreability of ρ, ν, and g). Hence, tests have to made without full dynamical similarity, and special attention must be given to the errors arising in the transfer of information to the full scale.

8

LOW AND HIGH REYNOLDS NUMBERS

8.1 Physical significance of the Reynolds number

The Reynolds number, introduced in the last chapter in the context of dynamical similarity, can be given a physical interpretation. This is useful in gaining an understanding of the dynamical processes that are important in different Reynolds number ranges, and in formulating corresponding approximations to the equations of motion.

To discuss this we need a name for each of the terms in the dynamical equation of steady incompressible flow:

$$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \mu \nabla^2 \boldsymbol{u}. \tag{8.1}$$

The second and third terms are given the obvious names pressure force and viscous force. The first term is called the inertia force. Physically, it is not a force, but it has the dimensions of force per unit volume and it is sometimes convenient to think of the dynamical equation in terms of a static balance between forces. The procedure is analogous to the more familiar use of the term centrifugal force to represent the acceleration involved in circular motion. No new idea is involved here, just a new name.

In the non-dimensional form of eqn (8.1),

$$\boldsymbol{u}' \cdot \nabla' \boldsymbol{u}' = -\nabla'(\Delta p)' + \frac{1}{\mathrm{Re}} \nabla'^2 \boldsymbol{u}' \tag{8.2}$$

(cf. eqn (7.13)), the primed quantities (possibly excepting $(\Delta p)'$) may be expected to be of order unity in magnitude. We shall see later that there are important qualifications to that statement. However, as a starting point it is justified so long as the length and velocity scales, U and L, have been chosen as typical quantities. Then a general speed will be of order U and $|\boldsymbol{u}'| \sim 1$; a general distance over which quantities vary significantly will be of order L and $\partial/\partial x'$, etc. will be of order unity.

Hence the ratio of the first term to the third in eqn (8.2) is of order Re. The corresponding terms in eqn (8.1) are in the same ratio. This indicates a physical interpretation of the Reynolds number as

$$\mathrm{Re} \sim \frac{\text{inertia forces}}{\text{viscous forces}}. \tag{8.3}$$

An alternative (entirely equivalent) formulation of this result, cited because we shall proceed in this way in subsequent chapters, is to write

$$|\boldsymbol{u}\cdot\nabla\boldsymbol{u}| \sim U^2/L, \quad |\nu\nabla^2\boldsymbol{u}| \sim \nu U/L^2. \tag{8.4}$$

Hence

$$\frac{|\boldsymbol{u}\cdot\nabla\boldsymbol{u}|}{|\nu\nabla^2\boldsymbol{u}|} \sim \frac{UL}{\nu} = \mathrm{Re}. \tag{8.5}$$

The Reynolds number thus indicates the relative importance of two dynamical processes. At a general point within the flow, the ratios of these two terms will not be exactly equal to the Reynolds number, but their characteristic magnitudes will be in this ratio.

8.2 Low Reynolds number

When the Reynolds number is much smaller than unity the viscous force dominates over the inertia force so much that the latter plays a negligible role in the flow dynamics. One may use an approximate form of the equation of motion with the inertia term dropped. Equation (8.2) becomes

$$0 = -\nabla'(\Delta p') + \frac{1}{\mathrm{Re}}\nabla'^2\boldsymbol{u}', \tag{8.6}$$

these terms being of order 1/Re and the neglected term of order 1. The pressure term must be retained since it is necessary to match the number of variables to the number of equations (Section 5.6). Physically, this means that the size of the pressure term is always governed by the other dynamically important terms—in this case by the viscous term.

Reverting to the dimensional form, eqn (8.6) is

$$\nabla p = \mu\nabla^2\boldsymbol{u}. \tag{8.7}$$

At every point in the fluid there is an effective balance between the local pressure and viscous forces. Equation (8.7) is known as the equation of creeping motion. It is evidently much simpler than the full Navier–Stokes equation, and solutions have been found for many cases for which the full equation has not yielded a solution. One case will be discussed in Section 9.4. Such solutions are found to agree well with the observed behaviour at low Reynolds number (see, e.g., Fig. 9.3), thus justifying the procedure leading to the approximation.

Two characteristic features of low Reynolds number flow are worth mentioning. Firstly, solutions of the equation of creeping motion are reversible; that is to say, if one has a solution, then there is another one

with the same streamline pattern but with the flow everywhere in the opposite direction (with all pressure gradients reversed). Hence, for example, the flow from right to left past an obstacle is the exact reverse of that from left to right. By extension, if one has an obstacle of a shape having upstream–downstream symmetry (its rear half is the mirror image of its front half), then the whole flow pattern has this symmetry; the pressure distribution is antisymmetric. We shall not derive these results formally, but it is readily seen that the solution to be presented in Section 9.4 possesses the above properties. We shall also be seeing some flows with this symmetry in Figs. 12.1, 12.6, and 12.7.

The second characteristic feature of low Reynolds number flows is that viscous interactions extend over large distances. For example, particles sedimenting at low Reynolds number affect each other's motion even when their separation is large compared with their size. Figure 8.1 illustrates this long-range viscous action for flow past a circular cylinder. It re-presents the information of Fig. 3.2 to show the velocity distribution across the mid-plane at a Reynolds number of 0.1. In the next section we shall be looking at the corresponding figure for high Reynolds number flow; comparison of the two provides a good illustration of the way in which different dynamical processes dominate in different Reynolds number ranges.

In this book further discussion of low Reynolds number flows will be confined to Sections 9.4 and 9.5 and parts of Chapter 12. There are

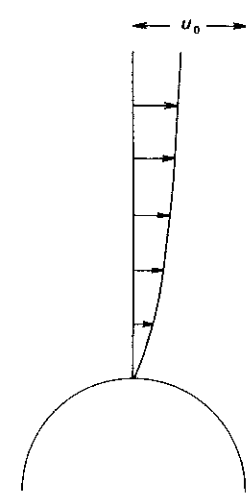

Fig. 8.1 Velocity distribution on centre plane in flow past circular cylinder at Re = 0.1.

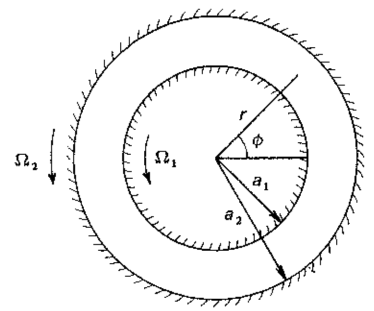

FIG. 9.1 Definition sketch for rotating Couette flow (z-axis is normal to paper).

assumptions, and the azimuthal and radial components of the Navier–Stokes equation become

$$0=\mu\left(\frac{d^2u_\phi}{dr^2}+\frac{1}{r}\frac{du_\phi}{dr}-\frac{u_\phi}{r^2}\right) \tag{9.3}$$

$$-\frac{\rho u_\phi^2}{r}=-\frac{dp}{dr} \tag{9.4}$$

with the boundary conditions

$$u_\phi=\Omega_1 a_1 \quad \text{at} \quad r=a_1; \quad u_\phi=\Omega_2 a_2 \quad \text{at} \quad r=a_2. \tag{9.5}$$

The first equation can be solved to give u_ϕ and this is then put into the second equation to give p. Thus, the distribution of the azimuthal velocity across the annulus is determined by the balance of viscous stresses, whilst the pressure distribution is determined by the balance between a radial pressure gradient and the centrifugal force associated with the circular motion.

The solution for u_ϕ (obtained by working in terms of the variable u_ϕ/r) is

$$u_\phi=Ar+B/r \tag{9.6}$$

where

$$A=(\Omega_2 a_2^2-\Omega_1 a_1^2)/(a_2^2-a_1^2), \quad B=(\Omega_1-\Omega_2)a_1^2a_2^2/(a_2^2-a_1^2). \tag{9.7}$$

The torque Σ_1 acting on the inner cylinder (per unit length in the z-direction) is given by the viscous stress† $\mu[r\partial(u_\phi/r)/\partial r]_{r=a_1}$ multiplied by the area $2\pi a_1$ and by the radius a_1; i.e.

$$\Sigma_1=4\pi\mu a_1^2a_2^2(\Omega_2-\Omega_1)/(a_2^2-a_1^2). \tag{9.8}$$

† That transformation to polar coordinates gives an expression of this form is to be expected from the fact that there will be no stress in rigid-body rotation, $u_\phi \propto r$.

Similarly, the torque on the outer cylinder

$$\Sigma_2=-4\pi\mu a_1^2a_2^2(\Omega_2-\Omega_1)/(a_2^2-a_1^2). \tag{9.9}$$

We notice that Σ_1 and Σ_2 are equal and opposite, as they must be since the total angular momentum of the fluid is not changing.

9.4 Stokes flow past a sphere

The most famous solution of the equation of creeping motion (8.7) applies to low Reynolds number flow past a sphere. It leads to the relationship between the velocity and the drag used to determine viscosity in the familiar procedure of observing the rate of fall of a sphere through a viscous fluid. This is often known as Stokes flow.

In spherical polar coordinates with $\theta=0$ in the flow direction in the frame of reference in which the sphere is at rest (Fig. 9.2) the equations are obtained from eqns (5.29)–(5.32) with the inertia terms (the left-hand sides of eqns (5.30)–(5.32)) and the body force terms put equal to zero, and with $u_\phi=0$, $\partial/\partial\phi=0$ by symmetry. The boundary conditions are

$$u_r=u_\theta=0 \quad \text{at} \quad r=a \tag{9.10}$$

$$u_r\to u_0\cos\theta, \quad u_\theta\to -u_0\sin\theta \quad \text{as} \quad r\to\infty \tag{9.11}$$

where u_0 is the free-stream velocity and a is the radius of the sphere. The solution is

$$u_r=u_0\cos\theta\left[1-\frac{3a}{2r}+\frac{a^3}{2r^3}\right] \tag{9.12}$$

$$u_\theta=-u_0\sin\theta\left[1-\frac{3a}{4r}-\frac{a^3}{4r^3}\right] \tag{9.13}$$

$$p-p_0=-\frac{3}{2}\frac{\mu u_0 a}{r^2}\cos\theta, \tag{9.14}$$

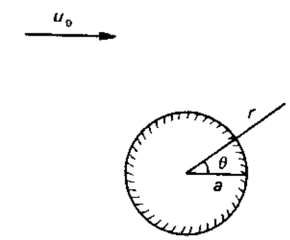

FIG. 9.2 Definition sketch for Stokes flow past a sphere (ϕ is azimuthal angle about $\theta=0$ axis).

p_0 being the ambient pressure. This may readily be shown by substituting the solution into the equations; for the forward integration the reader is referred to other sources (e.g. Refs. [11, 12]).

The force per unit area in the flow direction acting at a point on the surface of the sphere is the sum of the appropriate components of the viscous and pressure forces; that is†

$$\sigma = -\mu\left(\frac{\partial u_\theta}{\partial r}\right)_{r=a} \sin\theta - (p - p_0)_{r=a}\cos\theta. \tag{9.15}$$

Substituting the solution into this gives

$$\sigma = 3\mu u_0/2a. \tag{9.16}$$

Since this happens to be independent of θ the total force on the sphere is just σ multiplied by the surface area of the sphere,

$$D = 4\pi a^2\sigma = 6\pi\mu a u_0. \tag{9.17}$$

In terms of a drag coefficient defined by (7.16) with $L = 2a$, this is

$$C_D = 6\pi/\mathrm{Re} \tag{9.18}$$

(cf. (7.20)).

Equation (9.17) is the well-known result, due to Stokes, used in falling-sphere viscometry. Figure 9.3 shows a comparison of this result with experimental observations. The good agreement is important, not only for viscometry, but also because it demonstrates the validity of the reasoning leading to the equation of creeping motion and so encourages one to apply similar reasoning to other problems.

Figure 9.3 also shows the departures from Stokes's law that occur when the Reynolds number is too high for the creeping flow equation to apply. It is important to remember that eqn (9.17) can be used only when Re is less than about 0.5. (It is not sufficient, as is sometimes said, that the flow should be laminar.)

In applying this result it should also be remembered that, as remarked in Section 8.2, the viscous effects extend a long way at low Reynolds numbers. Distant boundaries may thus have a disturbingly large effect. In a falling-sphere viscometer, the container diameter must be more than one hundred times the sphere diameter for the error to be less than 2 per cent [192].

† The viscous stress in spherical polar co-ordinates reduces to this simple form for a boundary at rest.

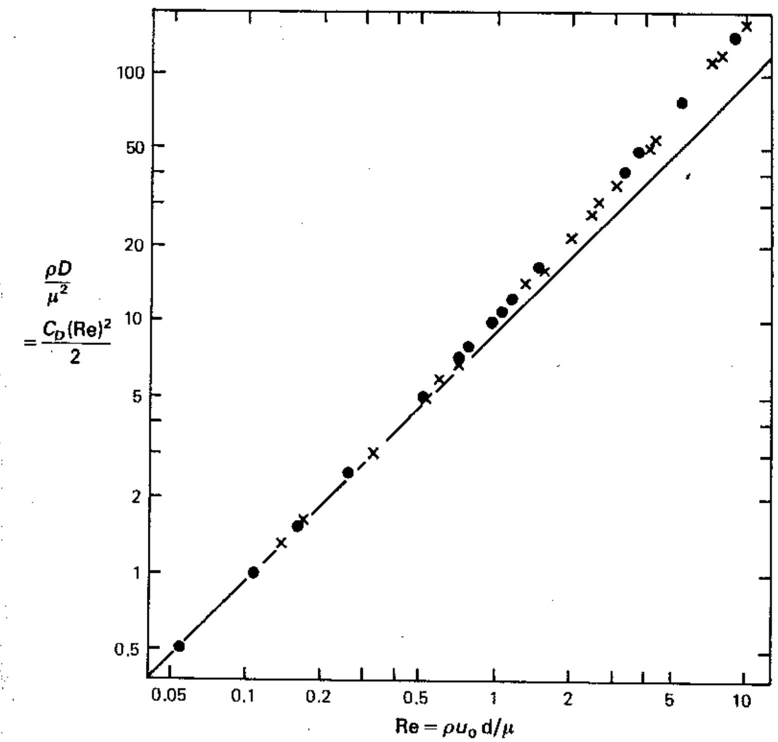

FIG. 9.3 Drag on a sphere at low Reynolds numbers. Experimental points from Refs. [248](×) and [336](●), both using the falling sphere method. The line represents eqn (9.17).

9.5 Low Reynolds number flow past a cylinder

Although the equation of creeping motion generally describes flow at low Reynolds number very satisfactorily, there is a complication that arises in two-dimensional flow, such as the flow past a circular cylinder. No solution of the equation can be found that matches to the boundary conditions both at the surface of the cylinder and at infinity (for mathematical details see, e.g. Ref. [11]). This fact is sometimes referred to as Stokes's paradox. Its resolution again involves the fact that the effect of a boundary on the velocity field extends to very large distances from that boundary. If we consider a moving cylinder in stationary ambient fluid, the fluid velocity remains comparable with the cylinder